A NONCOMMUTATIVE GEOMETRIC APPROACH TO THE QUANTUM STRUCTURE OF SPACETIME

R.B. ZHANG AND XIAO ZHANG

ABSTRACT. Together with collaborators, we introduced a noncommutative Riemannian geometry over Moyal algebras and systematically developed it for noncommutative spaces embedded in higher dimensions in the last few years. The theory was applied to construct a noncommutative version of general relativity, which is expected to capture some essential structural features of spacetime at the Planck scale. Examples of noncommutative spacetimes were investigated in detail. These include quantisations of plane-fronted gravitational waves, quantum Schwarzschild spacetime and Schwarzschild-de Sitter spacetime, and a quantun Tolman spacetime which is relevant to gravitational collapse. Here we briefly review the theory and its application in the study of quantum structure of spacetime.

CONTENTS

Key words and phrases. Noncommutative geometry, quantum spacetime, noncommutative general relativity, noncommutative Einstein field equations, Moyal algebra.

1. INTRODUCTION

1.1. **Quantum spacetime and noncommutative geometry.** It is a common consensus in the physics community that at the Planck scale (1.6×10^{-33} cm), quantum gravitational effects become dominant, and the usual notion of spacetime as a pseudo Riemannian manifold becomes obsolete. In the 40s, great masters like Heisenberg, Yang and others already pondered about the possibility that spacetime might become noncommutative [53, 62] at the Planck scale, and a rather convincing argument in support of this was given by Doplicher, Fredenhagen and Roberts [25] in the middle of 90s. The essence of their argument is as follows. When one localizes spacetime events with extreme precision, gravitational collapse will occur. Therefore, possible accuracy of localization of spacetime events should be limited in a quantum theory incorporating gravitation. This implies uncertainty relations for the different coordinates of spacetime events similar to Heisenberg's uncertainty principle in quantum mechanics. These authors then proposed a model of quantum spacetime based on a generalisation of the Moyal algebra, where the commutation relations among coordinates implement the uncertainty relations.

The argument of [25] is consistent with the current understanding of gravitational physics. It inevitably leads to the conclusion that some form of *noncommutative geometry* [20] will be necessary in order to describe the structure of spacetime at the Planck scale.

Early work of Yang [62] and of Snyder [53] already contained genesis of geometries which were not noncommutative, but noncommutative geometry became a mathematical subject in its own right only in the middle of 80s following Connes' work. Since then there has been much progress both in developing theories and exploring their applications. Many viewpoints were adopted and different mathematical approaches were followed by different researchers. Connes' theory [20] (see also [36]) formulated within the framework of C^*-algebras is the most successful, which incorporates cyclic cohomology and K-theory, and gives rise to noncommutative versions of index theorems. A central notion in the current formulation of Connes' theory is that of a spectral

triple involving an operator generalising the Dirac operator on a spin manifold. It is known that one can recover spin manifolds from the framework of spectral triples.

Theories generalizing aspects of algebraic geometry were also developed (see, e.g., [54] for a review and references). A notion of noncommutative schemes was formulated, which seems to provide a useful framework for developing noncommutative algebraic geometry. The seminal work of Kontsevich (see [43]) on deformation quantization [6] of Poisson manifolds and further developments [39] along a similar line are another aspect of the subject of noncommutative geometry.

1.2. **Applications of noncommutative geometry in quantum physics.** Noncommutative geometry has been applied to several areas in quantum physics. Connes and Lott [21] obtained the classical Lagrangian of the standard model by "dimensional reduction" from a noncommutative space to the four dimensional Minkowski space. Latter Chamseddine and Connes [16] used the more sophisticated method of spectral triples to incorporate gravity, which introduces additional interaction terms to the usual Lagrangian of standard model coupled to gravity. See [17] for recent developments.

Seiberg and Witten [52] showed that the anti-symmetric tensor field arising from massless states of strings can be described by the noncommutativity of a spacetime, where the algebra of functions is governed by the Moyal product. A considerable amount of research followed [52], see the review articles [26, 57] and references therein.

Noncommutative quantum field theoretical models were obtained by replacing the usual product of classical fields by the Moyal product (see [57] for a review and and references). Such theories were shown to have the curious property of mixing infrared and ultraviolet divergences in Feynman diagrams [47]. A solid result [38] in the area seems to be the proof of renormalisability of a noncommutative version of the ϕ^4 theory in 4-dimensions (which has an external harmonic oscillator potential term thus manifestly breaks translational invariance). However, many fundamental issues remain murky. For example, it was suggested that the Poincaré invariance of relativistic quantum field theory was deformed to a twisted Poincaré invariance [8, 11] in noncommutative field theory, but it appears rather impossible to define actions of the twisted Poincaré algebra on noncommutative fields [9]. There was even the surprising claim that noncommutative quantum field theories were completely equivalent to their undeformed counter parts [31].

An important application of noncommutative geometry is in the study of noncommutative generalisations of Einstein's theory of general relativity. A consistent formulation of a noncommutative version of general relativity is expected to give insight into a gravitational theory compatible with quantum mechanics. A unification of general relativity with quantum mechanics has long been sought after but remains as elusive as ever despite the extraordinary efforts put into string theory for the last three decades. The noncommutative geometrical approach may provide an alternative route. There

have been intensive research activities in this general direction inspired by [25]. We refer to [46] for a brief review. More references can also be found in [57].

1.3. **A noncommutative geometric approach to quantum spacetime.** In [13, 63], a noncommutative Riemannian geometry over Moyal algebras was developed, which retains key notions such as metric, connection and curvature of usual Riemannian geometry. The theory was applied to develop a noncommutative theory of general relativity, which is expected to capture essential features of quantum gravity.

We first quantised a space by deforming [34, 43] the algebra of functions to a noncommutative associative algebra, the Moyal algebra. Such an algebra naturally incorporates the generalised spacetime uncertainty relations of [25], capturing key features expected of spacetime at the Planck scale. We then systematically investigated the noncommutative Riemannian geometry of noncommutative spaces embedded in "higher dimensions". The general theory was extracted from the noncommutative Riemannian geometry of embedded spaces.

The theory of [13, 63] was first developed within a geometric framework analogous to the classical theory of embedded surfaces (see, e.g., [23]). This has the advantage of being concrete and explicit. Many examples of such noncommutative geometries can be easily constructed, which are transparently consistent in contrast. We then reformulated the theory algebraically [63] in terms of projective modules, a language commonly adopted in noncommutative geometry [20, 36]. Morally a projective module (a direct summand of a free module) is the geometric equivalent of an embedding of a low dimensional manifold isometrically in a higher dimensional one.

This connection between the algebraic notion of projective modules and the geometric notion of embeddings is particularly significant in view of Nash's isometric embedding theorem [48] and its generalisation to pseudo-Riemannian manifolds [32, 18, 37]. The theorems state that any (pseudo-) Riemannian manifold can be isometrically embedded in flat spaces. Therefore, in order to study the geometry of spacetime, one only needs to investigate (pseudo-) Riemannian manifolds embedded in higher dimensions. It is reasonable to anticipate something similar in the noncommutative setting. We should mention that embedded noncommutative spaces also play a role in the study of branes embedded in $\mathbb{R}^D$ in the context of Yang-Mills matrix models [55].

The theory of [13, 63] was applied to construct a noncommutative analogue of general relativity. In particular, a noncommutative Einstein field equation was proposed based on analysis of the noncommutative analogue of the second Bianchi identity. A new feature of the noncommutative field equation is the presence of another noncommutative Ricci curvature tensor (see (6.6)). The highly nonlinear nature of the equation makes it difficult to study, and noncommutativity adds further complication. Despite this, we have succeeded in constructing a class of exact solutions of the noncommutative Einstein equation, which are quantum deformations of plane-fronted gravitational waves [7, 29, 51, 30].

Quantisations of the Schwarzschild spacetime and Schwarzschild-de Sitter spacetime were also investigated within the framework of [13, 63]. They solve the noncommutative Einstein equation in the vacuum to the first order in the deformation parameter. However, higher order terms appear, which may be interpreted as matter sources. The physical origin and implications of the source terms are issues of physical interest. The Hawking temperature and entropy of the quantum Schwarzschild black hole were investigated, and a quantum correction to the entropy-area law was observed.

Gravitational collapse was also studied in the noncommutative setting [56]. Classically one may follow [59] to investigate the behaviour of the scalar curvature of the Tolman spacetime. When time increases to a certain critical value, the scalar curvature goes to infinity, thus the radius of the stellar object reduces to zero. By the reasoning of [59], this indicates gravitational collapse. [Obviously this only provides a snapshot of the evolution of the star, nevertheless, it enables one to gain some understanding of gravitational collapse.] A similar analysis in the noncommutative setting showed that gravitational collapse happens within a certain range of time instead of a single critical value, because of the quantum effects captured by the non-commutativity of spacetime. However, noncommutativity effect enters only at third order of the deformation parameter.

1.4. **The present paper.** This paper is a mathematical review of the theory developed in [13, 63] and its application to noncommutative gravity [60, 61, 56]. Its main body consists of two parts. One part comprises of Sections 2.1–5, which are all on the general theory, except for Section 4, where some elementary examples of noncommutative spaces are worked out in detail to illustrate the general theory. The other part comprises of Sections 6–7. In Section 6, we propose noncommutative Einstein field equations, and construct exact solutions for them in the vacuum. In section 7, we discuss the structure of the quantum versions of the Schwarzschild spacetime, Schwarzschild-de Sitter spacetime, and the Tolman spacetime which is relevant for gravitational collapse.

We emphasize that the present paper is nothing more than a streamlined account of the works [13, 63, 60, 61, 56]. It is certainly *not* meant to be a review of noncommutative general relativity. There is a vast body of literature in this subject area. The time and energy required to filter the literature to write an in-depth critique are beyond our means. As a result, we did not make any attempt to include all the references.

Here we merely comment that a variety of physically motivated methods and techniques were used to study corrections to general relativity arising from the noncommutativity of the Moyal algebra. For example, references [3, 4] studied deformations of the diffeomorphism algebra as a means for incorporating noncommutative effects of spacetime. [It is unfortunate that gravitational theories proposed this way [3, 4] were different [1] from the low energy limit of string theory.] In [12, 10] a gauge theoretical approached was taken. These approaches differ considerably from the theory of [13, 60, 61] mathematically. Other types of noncommutative Riemannian geometries

were also proposed [28, 22, 44, 45], which retain some of the familiar geometric notions like metric and curvature. Noncommutative analogues of the Hilbert-Einstein action were also suggested [14, 15] by treating noncommutative gravity as gauge theories. In the last couple of years, there have also been many other papers on noncommutative black holes, see e.g., [12, 2, 24, 5, 42]. We should also mention that Moyal planes have been treated from the point of view of spectral triples in [35].

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2. Differential geometry of noncommutative vector bundles

In this section we investigate general aspects of the noncommutative differential geometry over the Moyal algebra. We shall focus on the abstract theory here. A large class of examples will be given in later sections.

2.1. **Moyal algebra and projective modules.** We recall the definition of the Moyal algebra of smooth functions on an open region of $\mathbb{R}^n$, and also describe the finitely generated projective modules over the Moyal algebra. This provides the background material needed, and also serves to fix notation.

We take an open region U in $\mathbb{R}^n$ for a fixed n, and write the coordinate of a point $t \in U$ as $(t^1, t^2, \ldots, t^n)$. Let $\bar{h}$ be a real indeterminate, and denote by $\mathbb{R}[[\bar{h}]]$ the ring of formal power series in $\bar{h}$. Let $\mathcal{A}$ be the set of the formal power series in $\bar{h}$ with coefficients being real smooth functions on U. Namely, every element of $\mathcal{A}$ is of the form $\sum_{i\geq 0} f_i \bar{h}^i$ where f_i are smooth functions on U. Then $\mathcal{A}$ is an $\mathbb{R}[[\bar{h}]]$-module in the obvious way.

Fix a constant skew symmetric $n \times n$ matrix $\theta = (\theta_{ij})$. The Moyal product on $\mathcal{A}$ corresponding to θ is a map

$$\mu : \mathcal{A} \otimes_{\mathbb{R}[[\bar{h}]]} \mathcal{A} \longrightarrow \mathcal{A}, \quad f \otimes g \mapsto \mu(f,g),$$

defined by

$$\mu(f,g)(t) = \lim_{t' \to t} \exp^{\bar{h} \sum_{i,j} \theta_{ij} \frac{\partial}{\partial t^i} \frac{\partial}{\partial t'^j}} f(t) g(t'). \tag{2.1}$$

On the right hand side, $f(t)g(t')$ means the usual product of the functions f and g at t and t' respectively. Here $\exp^{\bar{h}\sum_{ij} \theta_{ij} \frac{\partial}{\partial t^i} \frac{\partial}{\partial t'^j}}$ should be understood as a power series in the differential operator $\sum_{i,j} \theta_{ij} \frac{\partial}{\partial t^i} \frac{\partial}{\partial t'^j}$. We extend μ $\mathbb{R}[[\bar{h}]]$-linearly to all elements in $\mathcal{A}$ by letting

$$\mu(\sum f_i \bar{h}^i, \sum g_j \bar{h}^j) := \sum \mu(f_i, g_j) \bar{h}^{i+j}.$$

It has been known since the late 40s from work of J. E. Moyal that the Moyal product is associative. Thus the $\mathbb{R}[[\bar{h}]]$-module $\mathcal{A}$ equipped with the Moyal product forms an associative algebra over $\mathbb{R}[[\bar{h}]]$, which is a deformation of the algebra of smooth

functions on U in the sense of [34]. We shall usually denote this associative algebra by $\mathcal{A}$, but when it is necessary to make explicit the multiplication, we shall write it as $(\mathcal{A},\mu)$.

The partial derivations $\partial_i := \frac{\partial}{\partial t^i}$ $(i = 1,2,\ldots,n)$ with respect to the coordinates t^i for U are $\mathbb{R}[[\bar{h}]]$-linear maps on $\mathcal{A}$. Since θ is a constant matrix, we have the Leibniz rule

$$\partial_i\mu(f,g) = \mu(\partial_i f,g)+\mu(f,\partial_i g) \tag{2.2}$$

for any elements f and g of $\mathcal{A}$. Therefore, the ∂_i are mutually commutating derivations of the Moyal algebra $(\mathcal{A},\mu)$.

Remark 2.1. The usual notation in the literature for $\mu(f,g)$ is $f*g$, which is also referred to as the star-product of f and g. Hereafter we shall replace μ by $*$ and simply write $\mu(f,g)$ as $f*g$.

Following the general philosophy of noncommutative geometry [20], we regard the noncommutative associative algebra $(\mathcal{A},\mu)$ as defining some *quantum deformation of the region* U. Finitely generated projective modules over $\mathcal{A}$ will be regarded as (spaces of sections of) noncommutative vector bundles on the quantum deformation of U defined by $\mathcal{A}$. Let us now briefly describe finitely generated projective $\mathcal{A}$-modules.

Given an integer $m > n$, we let ${}_l\mathcal{A}^m$ (resp. $\mathcal{A}^m_r$) be the set of m-tuples with entries in $\mathcal{A}$ written as rows (resp. columns). We shall regard ${}_l\mathcal{A}^m$ (resp. $\mathcal{A}^m_r$) as a left (resp. right) $\mathcal{A}$-module with the action defined by multiplication from the left (resp. right). More explicitly, for $v = \begin{pmatrix} a_1 & a_2 & \ldots & a_m\end{pmatrix} \in {}_l\mathcal{A}^m$, and $b \in \mathcal{A}$, we have $b*v = \begin{pmatrix} b*a_1 & b*a_2 & \ldots & b*a_m\end{pmatrix}$. Similarly for $w = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m\end{pmatrix} \in \mathcal{A}^m_r$, we have $w*b = \begin{pmatrix} a_1*b \\ a_2*b \\ \vdots \\ a_m*b\end{pmatrix}$. Let $\mathbf{M}_m(\mathcal{A})$ be the set of $m\times m$-matrices with entries in $\mathcal{A}$. We define matrix multiplication in the usual way but by using the Moyal product for products of matrix entries, and still denote the corresponding matrix multiplication by $*$. Now for $A = (a_{ij})$ and $B = (b_{ij})$, we have $(A*B) = (c_{ij})$ with $c_{ij} = \sum_k a_{ik}*b_{kj}$. Then $\mathbf{M}_m(\mathcal{A})$ is an $\mathbb{R}[[h]]$-algebra, which has a natural left (resp. right) action on $\mathcal{A}^m_r$ (resp. ${}_l\mathcal{A}^m$).

A finitely generated projective left (reps. right) $\mathcal{A}$-module is isomorphic to some direct summand of ${}_l\mathcal{A}^m$ (resp. $\mathcal{A}^m_r$) for some $m < \infty$. If $e \in \mathbf{M}_m(\mathcal{A})$ satisfies the condition $e*e = e$, that is, it is an idempotent, then

$$\mathcal{M} = {}_l\mathcal{A}^m * e := \{v*e \mid v \in {}_l\mathcal{A}^m\}, \quad \tilde{\mathcal{M}} = e*\mathcal{A}^m_r := \{e*w \mid \in \mathcal{A}^m_r\}$$

are respectively projective left and right $\mathcal{A}$-modules. Furthermore, every projective left (right) $\mathcal{A}$-module is isomorphic to an $\mathcal{M}$ (resp. $\tilde{\mathcal{M}}$) constructed this way by using some idempotent e.

2.2. **Connections and curvatures.** We start by considering the action of the partial derivations ∂_i on $\mathcal{M}$ and $\tilde{\mathcal{M}}$. We only treat the left module in detail, and present the pertinent results for the right module at the end, since the two cases are similar.

Let us first specify that ∂_i acts on rectangular matrices with entries in $\mathcal{A}$ by componentwise differentiation. More explicitly,

$$\partial_i B = \begin{pmatrix} \partial_i b_{11} & \partial_i b_{12} & \ldots & \partial_i b_{1l} \\ \partial_i b_{21} & \partial_i b_{22} & \ldots & \partial_i b_{2l} \\ \ldots & \ldots & \ldots & \ldots \\ \partial_i b_{k1} & \partial_i b_{k2} & \ldots & \partial_i b_{kl} \end{pmatrix} \quad \text{for} \quad B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1l} \\ b_{21} & b_{22} & \ldots & b_{2l} \\ \ldots & \ldots & \ldots & \ldots \\ b_{k1} & b_{k2} & \ldots & b_{kl} \end{pmatrix}.$$

In particular, given any $\zeta = v * e \in \mathcal{M}$, where $v \in {}_l\mathcal{A}^m$ regarded as a row matrix, we have $\partial_i \zeta = (\partial_i v) * e + v * \partial_i(e)$ by the Leibniz rule. While the first term belongs to $\mathcal{M}$, the second term does not in general. Therefore, ∂_i $(i = 1, 2, \ldots, n)$ send $\mathcal{M}$ to some subspace of ${}_l\mathcal{A}^m$ different from $\mathcal{M}$.

Let $\omega_i \in \mathbf{M}_m(\mathcal{A})$ $(i = 1, 2, \ldots, n)$ be $m \times m$-matrices with entries in $\mathcal{A}$ satisfying the following condition:

$$e * \omega_i * (1 - e) = -e * \partial_i e, \quad \forall i. \tag{2.3}$$

Define the $\mathbb{R}[[\bar{h}]]$-linear maps ∇_i $(i = 1, 2, \ldots, n)$ from $\mathcal{M}$ to ${}_l\mathcal{A}^m$ by

$$\nabla_i \zeta = \partial_i \zeta + \zeta * \omega_i, \quad \forall \zeta \in \mathcal{M}.$$

Then each ∇_i is a covariant derivative on the noncommutative bundle $\mathcal{M}$ in the sense of Theorem 2.2 below. They together define a *connection* on $\mathcal{M}$.

Theorem 2.2. *The maps ∇_i $(i = 1, 2, \ldots, n)$ have the following properties. For all $\zeta \in \mathcal{M}$ and $a \in \mathcal{A}$,*

$$\nabla_i \zeta \in \mathcal{M} \quad \textit{and} \quad \nabla_i(a * \zeta) = \partial_i(a) * \zeta + a * \nabla_i \zeta.$$

Proof. For any $\zeta \in \mathcal{M}$, we have

$$\begin{aligned} \nabla_i(\zeta) * e &= \partial_i(\zeta) * e + \zeta * \omega_i * e \\ &= \partial_i \zeta + \zeta * (\omega_i * e - \partial_i e), \end{aligned}$$

where we have used the Leibniz rule and also the fact that $\zeta * e = \zeta$. Using this latter fact again, we have $\zeta * (\omega_i * e - \partial_i e) = \zeta * (e * \omega_i * e - e * \partial_i e)$, and by the defining property (2.3) of ω_i, we obtain $\zeta * (e * \omega_i * e - e * \partial_i * e) = \zeta * \omega_i$. Hence

$$\nabla_i(\zeta) * e = \partial_i \zeta + \zeta * \omega_i = \nabla_i \zeta,$$

proving that $\nabla_i \zeta \in \mathcal{M}$. The second part of the theorem immediately follows from the Leibniz rule. □

We shall also say that the set of ω_i ($i = 1,2,\dots,n$) is a connection on $\mathcal{M}$. Since $e * \partial_i e = \partial_i(e) * (1-e)$, one obvious choice for ω_i is $\omega_i = -\partial_i e$, which we shall refer to as the *canonical connection* on $\mathcal{M}$.

The following result is an easy consequence of (2.3).

Lemma 2.3. *If ω_i ($i = 1,2,\dots,n$) define a connection on $\mathcal{M}$, then so do also $\omega_i + \phi_i * e$ ($i = 1,2,\dots,n$) for any $m \times m$-matrices ϕ_i with entries in $\mathcal{A}$.*

For a given connection ω_i ($i = 1,2,\dots,n$), we consider $[\nabla_i, \nabla_j] = \nabla_i\nabla_j - \nabla_j\nabla_i$ with the right hand side understood as composition of maps on $\mathcal{M}$. Simple calculations show that for all $\zeta \in \mathcal{M}$,

$$[\nabla_i, \nabla_j]\zeta = \zeta * \mathcal{R}_{ij} \quad \text{with} \quad \mathcal{R}_{ij} := \partial_i\omega_j - \partial_j\omega_i - [\omega_i,\ \omega_j]_*,$$

where $[\omega_i, \omega_j]_* = \omega_i * \omega_j - \omega_j * \omega_i$ is the commutator. We call $\mathcal{R}_{ij}$ the *curvature* of $\mathcal{M}$ associated with the connection ω_i.

For all $\zeta \in \mathcal{M}$,

$$\begin{aligned} [\nabla_i, \nabla_j]\nabla_k\zeta &= \partial_k(\zeta) * \mathcal{R}_{ij} + \zeta * \omega_k * \mathcal{R}_{ij}, \\ \nabla_k[\nabla_i, \nabla_j]\zeta &= \partial_k(\zeta) * \mathcal{R}_{ij} + \zeta * (\partial_k\mathcal{R}_{ij} + \mathcal{R}_{ij} * \omega_k). \end{aligned}$$

Define the following covariant derivatives of the curvature:

$$\nabla_k\mathcal{R}_{ij} := \partial_k\mathcal{R}_{ij} + \mathcal{R}_{ij} * \omega_k - \omega_k * \mathcal{R}_{ij}, \tag{2.4}$$

we have

$$[\nabla_k, [\nabla_i, \nabla_j]]\zeta = \zeta * \nabla_k\mathcal{R}_{ij}, \quad \forall\zeta \in \mathcal{M}.$$

The Jacobian identity $[\nabla_k, [\nabla_i, \nabla_j]] + [\nabla_j, [\nabla_k, \nabla_i]] + [\nabla_i, [\nabla_j, \nabla_k]] = 0$ leads to

$$\zeta * (\nabla_k\mathcal{R}_{ij} + \nabla_j\mathcal{R}_{ki} + \nabla_i\mathcal{R}_{jk}) = 0, \quad \forall\zeta \in \mathcal{M}.$$

From this we immediately see that $e * (\nabla_k\mathcal{R}_{ij} + \nabla_j\mathcal{R}_{ki} + \nabla_i\mathcal{R}_{jk}) = 0$. In fact, the following stronger result holds.

Theorem 2.4. *The curvature satisfies the following* Bianchi identity*:*

$$\nabla_k\mathcal{R}_{ij} + \nabla_j\mathcal{R}_{ki} + \nabla_i\mathcal{R}_{jk} = 0.$$

Proof. The proof is entirely combinatorial. Let

$$\begin{aligned} A_{ijk} &= \partial_k\partial_i\omega_j - \partial_k\partial_j\omega_i, \\ B_{ijk} &= [\partial_i\omega_j, \omega_k]_* - [\partial_j\omega_i, \omega_k]_*. \end{aligned}$$

Then we can express $\nabla_k\mathcal{R}_{ij}$ as

$$\nabla_k\mathcal{R}_{ij} = A_{ijk} + B_{ijk} - \partial_k[\omega_i,\ \omega_j]_* - [[\omega_i,\ \omega_j]_*,\ \omega_k]_*.$$

Note that

$$\begin{aligned} A_{ijk} + A_{jki} + A_{kij} &= 0, \\ B_{ijk} + B_{jki} + B_{kij} &= \partial_k[\omega_i,\ \omega_j]_* + \partial_i[\omega_j,\ \omega_k]_* + \partial_j[\omega_k,\ \omega_i]_*. \end{aligned}$$

Using these relations together with the Jacobian identity

$$[[\omega_i,\ \omega_j]_*,\ \omega_k]_* + [[\omega_j,\ \omega_k]_*,\ \omega_i]_* + [[\omega_k,\ \omega_i]_*,\ \omega_j]_* = 0,$$

we easily prove the Bianchi identity. □

2.3. **Gauge transformations.** Let $GL_m(\mathcal{A})$ be the group of invertible $m\times m$-matrices with entries in $\mathcal{A}$. Let $\mathcal{G}$ be the subgroup defined by

$$\mathcal{G} = \{g \in GL_m(\mathcal{A}) \mid e*g = g*e\}, \tag{2.5}$$

which will be referred to as the *gauge group*. There is a right action of $\mathcal{G}$ on $\mathcal{M}$ defined, for any $\zeta\in\mathcal{M}$ and $g\in\mathcal{G}$, by $\zeta\times g\mapsto \zeta\cdot g := \zeta * g$, where the right side is defined by matrix multiplication. Clearly, $\zeta*g*e = \zeta*g$. Hence $\zeta*g\in\mathcal{M}$, and we indeed have a $\mathcal{G}$ action on $\mathcal{M}$.

For a given $g\in\mathcal{G}$, let

$$\omega_i^g = g^{-1}*\omega_i*g - g^{-1}*\partial_i g. \tag{2.6}$$

Then

$$e*\omega_i^g*(1-e) = g^{-1}*e*\omega_i*(1-e)*g - g^{-1}*e*\partial_i(g)*(1-e).$$

By (2.3), $g^{-1}*e*\omega_i*(1-e)*g = -g^{-1}*e*\partial_i(e)*g$. Using the defining property of the gauge group $\mathcal{G}$, we can show that

$$g^{-1}*e*\omega_i*(1-e)*g = -e*\partial_i e + g^{-1}*e*\partial_i(g)*(1-e).$$

Therefore, $e*\omega_i^g*(1-e) = -e*\partial_i e$. This shows that the ω_i^g satisfy the condition (2.3), thus form a connection on $\mathcal{M}$.

Now for any given $g\in\mathcal{G}$, define the maps ∇_i^g on $\mathcal{M}$ by

$$\nabla_i^g\zeta = \partial_i\zeta + \zeta*\omega_i^g, \quad \forall\zeta.$$

Also, let $\mathcal{R}_{ij}^g = \partial_i\omega_j^g - \partial_j\omega_i^g - [\omega_i^g, \omega_j^g]_*$ be the curvature corresponding to the connection ω_i^g. Then we have the following result.

Lemma 2.5. *Under a gauge transformation procured by $g\in\mathcal{G}$,*

$$\nabla_i^g(\zeta*g) = \nabla_i(\zeta)*g, \quad \forall\zeta\in\mathcal{M};$$
$$\mathcal{R}_{ij}^g = g^{-1}*\mathcal{R}_{ij}*g.$$

Proof. Note that

$$\nabla_i^g(\zeta*g) = \partial_i(\zeta)*g + \zeta*\partial_i g + \zeta*g*\omega_i^g = (\partial_i\zeta + \zeta*\omega_i)*g.$$

This proves the first formula.

To prove the second claim, we use the following formulae

$$\begin{aligned}\partial_i\omega_j^g - \partial_j\omega_i^g &= g^{-1} * (\partial_i\omega_j - \partial_j\omega_i) * g - \partial_i(g^{-1}) * \partial_j g + \partial_j(g^{-1}) * \partial_i g \\ &\quad + [\partial_i(g^{-1}) * g,\ g^{-1} * \omega_j * g]_* - [\partial_j(g^{-1}) * g,\ g^{-1} * \omega_i * g]_*; \\ [\omega_i^g,\ \omega_j^g]_* &= g^{-1} * [\omega_i,\ \omega_j]_* * g - \partial_i(g^{-1}) * \partial_j g + \partial_j(g^{-1}) * \partial_i g \\ &\quad + [\partial_i(g^{-1}) * g,\ g^{-1} * \omega_j * g]_* - [\partial_j(g^{-1}) * g,\ g^{-1} * \omega_i * g]_*.\end{aligned}$$

Combining these formulae together we obtain $\mathcal{R}_{ij}^g = g^{-1}\mathcal{R}_{ij}g$. This completes the proof of the lemma. □

2.4. **Vector bundles associated to right projective modules.** Connections and curvatures can be introduced for the right bundle $\tilde{\mathcal{M}} = e * \mathcal{A}_r^m$ in much the same way. Let $\tilde{\omega}_i \in \mathbf{M}_m(\mathcal{A})$ $(i = 1, 2, \ldots, n)$ be matrices satisfying the condition that

$$(1 - e) * \tilde{\omega}_i * e = \partial_i(e) * e. \tag{2.7}$$

Then we can introduce a connection consisting of the right covariant derivatives $\tilde{\nabla}_i$ $(i = 1, 2, \ldots, n)$ on $\tilde{\mathcal{M}}$ defined by

$$\tilde{\nabla}_i : \tilde{\mathcal{M}} \longrightarrow \tilde{\mathcal{M}}, \qquad \xi \mapsto \tilde{\nabla}_i\xi = \partial_i\xi - \tilde{\omega}_i * \xi.$$

It is easy to show that $\tilde{\nabla}_i(\xi * a) = \tilde{\nabla}_i(\xi) * a + \xi * \partial_i a$ for all $a \in \mathcal{A}$.

Note that if $\tilde{\omega}_i = \partial_i e$ for all i, the condition (2.7) is satisfied. We call them the *canonical connection* on $\tilde{\mathcal{M}}$.

Returning to a general connection $\tilde{\omega}_i$, we define the associated curvature by

$$\tilde{\mathcal{R}}_{ij} = \partial_i\tilde{\omega}_j - \partial_j\tilde{\omega}_i - [\tilde{\omega}_i,\ \tilde{\omega}_j]_*.$$

Then for all $\xi \in \tilde{\mathcal{M}}$, we have

$$[\tilde{\nabla}_i,\ \tilde{\nabla}_j]\xi = -\tilde{\mathcal{R}}_{ij} * \xi.$$

We further define the covariant derivatives of $\tilde{\mathcal{R}}_{ij}$ by

$$\tilde{\nabla}_k\tilde{\mathcal{R}}_{ij} = \partial_k\tilde{\mathcal{R}}_{ij} + \tilde{\omega}_k * \tilde{\mathcal{R}}_{ij} - \tilde{\mathcal{R}}_{ij} * \tilde{\omega}_k.$$

Then we have the following result.

Lemma 2.6. *The curvature on the right bundle $\tilde{\mathcal{M}}$ satisfies the Bianchi identity*

$$\tilde{\nabla}_i\tilde{\mathcal{R}}_{jk} + \tilde{\nabla}_j\tilde{\mathcal{R}}_{ki} + \tilde{\nabla}_k\tilde{\mathcal{R}}_{ij} = 0.$$

By direct calculations we can also prove the following result:

$$[\tilde{\nabla}_k, [\tilde{\nabla}_i,\ \tilde{\nabla}_j]]\xi = -\tilde{\nabla}_k(\tilde{\mathcal{R}}_{ij}) * \xi, \quad \forall \xi \in \tilde{\mathcal{M}}.$$

Consider the gauge group $\mathcal{G}$ defined by (2.5), which has a right action on $\tilde{\mathcal{M}}$:

$$\tilde{\mathcal{M}} \times \mathcal{G} \longrightarrow \tilde{\mathcal{M}}, \quad \xi \times g \mapsto \xi \cdot g := g^{-1} * \xi.$$

Under a gauge transformation procured by $g \in \mathcal{G}$,

$$\tilde{\omega}_i \mapsto \tilde{\omega}_i^g := g^{-1} * \tilde{\omega}_i * g + \partial_i(g^{-1}) * g.$$

The connection $\tilde{\nabla}_i^g$ on $\tilde{\mathcal{M}}$ defined by

$$\tilde{\nabla}_i^g \xi = \partial_i \xi - \tilde{\omega}_i^g * \xi$$

satisfies the following relation for all $\xi \in \tilde{\mathcal{M}}$:

$$\tilde{\nabla}_i^g (g^{-1} * \xi) = g^{-1} * \tilde{\nabla}_i \xi.$$

Furthermore, the gauge transformed curvature

$$\tilde{\mathcal{R}}_{ij}^g := \partial_i \tilde{\omega}_j^g - \partial_j \tilde{\omega}_i^g - [\tilde{\omega}_i^g, \tilde{\omega}_j^g]_*$$

is related to $\tilde{\mathcal{R}}_{ij}$ by

$$\tilde{\mathcal{R}}_{ij}^g = g^{-1} * \tilde{\mathcal{R}}_{ij} * g.$$

Given any $\Lambda \in \mathbf{M}_m(\mathcal{A})$, we can define the $\mathcal{A}$-bimodule map

$$\langle\ ,\ \rangle : \mathcal{M} \otimes_{\mathbb{R}[[\bar{h}]]} \tilde{\mathcal{M}} \longrightarrow \mathcal{A}, \quad \zeta \otimes \xi \mapsto \langle \zeta, \xi \rangle = \zeta * \Lambda * \xi, \tag{2.8}$$

where $\zeta * \Lambda * \xi$ is defined by matrix multiplication. We shall say that the bimodule homomorphism is *gauge invariant* if for any element g of the gauge group $\mathcal{G}$,

$$\langle \zeta \cdot g, \xi \cdot g \rangle = \langle \zeta, \xi \rangle, \quad \forall \zeta \in \mathcal{M},\ \xi \in \tilde{\mathcal{M}}.$$

Also, the bimodule homomorphism is said to be *compatible with the connections* ω_i on $\mathcal{M}$ and $\tilde{\omega}_i$ on $\tilde{\mathcal{M}}$ if for all $i = 1, 2, \ldots, n$

$$\partial_i \langle \zeta, \xi \rangle = \langle \nabla_i \zeta, \xi \rangle + \langle \zeta, \tilde{\nabla}_i \xi \rangle, \quad \forall \zeta \in \mathcal{M},\ \xi \in \tilde{\mathcal{M}}.$$

Lemma 2.7. *Let $\langle\ ,\ \rangle : \mathcal{M} \otimes_{\mathbb{R}[[\bar{h}]]} \tilde{\mathcal{M}} \longrightarrow \mathcal{A}$ be an $\mathcal{A}$-bimodule homomorphism defined by* (2.8) *with a given $m \times m$-matrix Λ with entries in $\mathcal{A}$. Then*

(1) $\langle\ ,\ \rangle$ *is gauge invariant if $g * \Lambda * g^{-1} = \Lambda$ for all $g \in \mathcal{G}$;*
(2) $\langle\ ,\ \rangle$ *is compatible with the connections ω_i on $\mathcal{M}$ and $\tilde{\omega}_i$ on $\tilde{\mathcal{M}}$ if for all i,*

$$e * (\partial_i \Lambda - \omega_i * \Lambda + \Lambda * \tilde{\omega}_i) * e = 0.$$

Proof. Note that $\langle \zeta \cdot g, \xi \cdot g \rangle = \zeta * g * \Lambda * g^{-1} * \xi$ for any $g \in \mathcal{G}$, $\zeta \in \mathcal{M}$ and $\xi \in \tilde{\mathcal{M}}$. Therefore $\langle \zeta \cdot g, \xi \cdot g \rangle = \langle \zeta, \xi \rangle$ if $g * \Lambda * g^{-1} = \Lambda$. This proves part (1).

Now $\partial_i \langle \zeta, \xi \rangle = \langle \partial_i \zeta, \xi \rangle + \langle \zeta, \partial_i \xi \rangle + \zeta * (\partial_i \Lambda - \omega_i * \Lambda + \Lambda * \tilde{\omega}_i) * \xi$. Thus if Λ satisfies the condition of part (2), then $\langle\ ,\ \rangle$ is compatible with the connections. $\square$

2.5. **Canonical connections and fibre metric.** Let us consider in detail the canonical connections on $\mathcal{M}$ and $\tilde{\mathcal{M}}$ given by

$$\omega_i = -\partial_i e, \quad \tilde{\omega}_i = \partial_i e.$$

A particularly nice feature in this case is that the corresponding curvatures on the left and right bundles coincide. We have the following formula:

$$\mathcal{R}_{ij} = \tilde{\mathcal{R}}_{ij} = -[\partial_i e,\ \partial_j e]_*. \tag{2.9}$$

Now we consider a special case of the $\mathcal{A}$-bimodule map defined by equation (2.8).

Definition 2.8. Denote by $\mathbf{g} : \mathcal{M} \otimes_{\mathbb{R}[[\bar{h}]]} \tilde{\mathcal{M}} \longrightarrow \mathcal{A}$ the map defined by (2.8) with Λ being the identity matrix. We shall call $\mathbf{g}$ the *fibre metric* on $\mathcal{M}$.

Lemma 2.9. *The fibre metric* $\mathbf{g}$ *is gauge invariant and is compatible with the standard connections.*

Proof. Since Λ is the identity matrix in the present case, it immediately follows from Lemma 2.7 (1) that $\mathbf{g}$ is gauge invariant. Note that $e * \partial_i(e) * e = 0$ for all i. Using this fact in Lemma 2.7 (2), we easily see that $\mathbf{g}$ is compatible with the standard connections. □

3. EMBEDDED NONCOMMUTATIVE SPACES

In [13], we introduced noncommutative spaces which are embedded in "higher dimensions" in a geometric setting. Here we recall this theory and reformulate it in the framework of Section 2 in terms of projective modules. This also provides a class of explicit examples of idempotents and related projective modules.

3.1. **Embedded noncommutative spaces.** We shall consider only embedded spaces with Euclidean signature. The Minkowski case is similar, which will be briefly discussed in Section 3.3. Given $X = \begin{pmatrix} X^1 & X^2 & \dots & X^m \end{pmatrix}$ in ${}_l\mathcal{A}^m$, we define an $n \times n$ matrix $\mathbf{g} = (g_{ij})_{i,j=1,2,\dots,n}$ with entries given by

$$g_{ij} = \sum_{\alpha=1}^{m} \partial_i X^\alpha * \partial_j X^\alpha.$$

Let $g^0 = \mathbf{g} \mod \bar{h}$, which is an $n \times n$-matrix of smooth functions on U. Assume that g^0 is invertible at every point $t \in U$. Then there exists a unique $n \times n$-matrix $\left(g^{ij}\right)$ over $\mathcal{A}$ which is the right inverse of $\mathbf{g}$, i.e.,

$$g_{ij} * g^{jk} = \delta_i^k,$$

where we have used Einstein's convention of summing over repeated indices. To see this, we need to examine the definition (2.1) of the Moyal product more carefully. Let

$$\mu_p : \mathcal{A}/\bar{h}\mathcal{A} \otimes \mathcal{A}/\bar{h}\mathcal{A} \longrightarrow \mathcal{A}/\bar{h}\mathcal{A}, \quad p = 0, 1, 2, \dots, \tag{3.1}$$

be $\mathbb{R}$-linear maps defined by

$$\mu_p(f,g) = \lim_{t'\to t} \frac{1}{p!}\left(\sum_{ij} \theta_{i,j}\frac{\partial}{\partial t^i}\frac{\partial}{\partial t'^j}\right)^p f(t)g(t').$$

Then $f*g = \sum_{p=0}^{\infty} \bar{h}^p \mu_p(f,g)$. Now write $g_{ij} = \sum_p \bar{h}^p g_{ij}[p]$ and $g^{ij} = \sum_p \bar{h}^p \tilde{g}^{ij}[p]$, where $(g^{ij}[0])$ is the inverse of $(g_{ij}[0])$. Now in terms of the maps μ_k defined by (3.1), we have

$$\delta_i^k = g_{ij} * g^{jk} = \sum_q \bar{h}^q \sum_{m+n+p=q} \mu_p(g_{ij}[m], g^{jk}[n]),$$

which is equivalent to

$$g^{ij}[q] = -\sum_{n=1}^{q}\sum_{m=0}^{q-n} g^{ik}[0]\mu_n(g_{kl}[m], g^{lj}[q-n-m]).$$

Since the right-hand side involves only $g^{lj}[r]$ with $r<q$, this equation gives a recursive formula for the right inverse of $\mathbf{g}$. In the same way, we can also show that there also exists a unique left inverse of $\mathbf{g}$. It follows from the associativity of multiplication of matrices over any associative algebra that the left and right inverses of $\mathbf{g}$ are equal.

It is easy to see that if $\mathbf{g}$ is invertible, then g^0 is nonsingular.

Definition 3.1. We call an element $X \in {}_l\mathcal{A}^m$ an *embedded noncommutative space* if g^0 is invertible for all $t \in U$. In this case, $\mathbf{g}$ is called the *metric* of the noncommutative space.

Let

$$E_i = \partial_i X, \quad \tilde{E}^i = (E_j)^t * g^{ji}, \quad E^i = g^{ij} * E_j,$$

for $i = 1,2,\ldots,n$, where $(E_i)^t = \begin{pmatrix} \partial_i X^1 \\ \partial_i X^2 \\ \vdots \\ \partial_i X^m \end{pmatrix}$ denotes the transpose of E_i. Define $e \in \mathbf{M}_m(\mathcal{A})$ by

$$\begin{aligned} e &:= \tilde{E}^j * E_j \\ &= \begin{pmatrix} \partial_i X^1 * g^{ij} * \partial_j X^1 & \partial_i X^1 * g^{ij} * \partial_j X^2 & \ldots & \partial_i X^1 * g^{ij} * \partial_j X^m \\ \partial_i X^2 * g^{ij} * \partial_j X^1 & \partial_i X^2 * g^{ij} * \partial_j X^2 & \ldots & \partial_i X^2 * g^{ij} * \partial_j X^m \\ \ldots & \ldots & \ldots & \ldots \\ \partial_i X^m * g^{ij} * \partial_j X^1 & \partial_i X^m * g^{ij} * \partial_j X^2 & \ldots & \partial_i X^m * g^{ij} * \partial_j X^m \end{pmatrix}. \end{aligned} \tag{3.2}$$

We have the following results.

Proposition 3.2. (1) *Under matrix multiplication,* $E_i * \tilde{E}^j = \delta_i^j$ *for all* i *and* j.

(2) *The* $m \times m$ *matrix* e *satisfies* $e*e = e$, *that is, it is an idempotent in* $\mathbf{M}_m(\mathcal{A})$.

(3) *The left and right projective $\mathcal{A}$-modules $\mathcal{M} = {}_l\mathcal{A}^m * e$ and $\tilde{\mathcal{M}} = e * \mathcal{A}_r^m$ are respectively spanned by E_i and $\tilde{E}^i$. More precisely, we have*

$$\mathcal{M} = \{a^i * E_i \mid a^i \in \mathcal{A}\}, \quad \tilde{\mathcal{M}} = \{\tilde{E}^i * b_i \mid b_i \in \mathcal{A}\}.$$

Proof. Note that $g_{ij} = E_i * (E_j)^t$. Thus $E_i * \tilde{E}^j = E_i * (E_k)^t * g^{kj} = \delta_i^j$. It then immediately follows that

$$e * e = \tilde{E}^i * \left(E_i * \tilde{E}^j\right) * E_j = \tilde{E}^i * \delta_i^j * E_j = e.$$

Obviously $\mathcal{M} \subset \{a^i * E_i \mid a^i \in \mathcal{A}\}$ and $\tilde{\mathcal{M}} \subset \{\tilde{E}^i * b_i \mid b_i \in \mathcal{A}\}$. By the first part of the proposition, we have

$$a^i * E_i * e = a^i * \left(E_i * \tilde{E}^j\right) * E_j = a^j * E_j,$$
$$e * \tilde{E}^j * b_j = \tilde{E}^i * \left(E_i * \tilde{E}^j\right) * b_j = \tilde{E}^i * b_i.$$

This proves the last claim of the proposition. □

It is also useful to observe that $\tilde{\mathcal{M}} = \{(E_i)^t * b_i \mid b_i \in \mathcal{A}\}$ since $\mathbf{g}$ is invertible.

We shall denote $\mathcal{M}$ and $\tilde{\mathcal{M}}$ respectively by TX and $\tilde{T}X$, and refer to them as the *left* and *right tangent bundles* of the noncommutative space X. Note that the definition of the tangent bundles coincides with that in [13].

The proposition below in particular shows that the fibre metric $\mathbf{g} : TX \otimes_{\mathbb{R}[[\bar{h}]]} \tilde{T}X \longrightarrow \mathcal{A}$ defined in Definition 2.8 agrees with the metric of the embedded noncommutative space defined in Definition 3.1.

Proposition 3.3. *For any $\zeta = a^i * E_i \in TX$ and $\xi = (E_j)^t * b^j \in \tilde{T}X$ with $a_i, b_j \in \mathcal{A}$,*

$$\mathbf{g} : \zeta \otimes \xi \mapsto \mathbf{g}(\zeta, \xi) = a^i * g_{ij} * b^j.$$

In particular, $\mathbf{g}(E_i, (E_j)^t) = g_{ij}$.

Proof. Recall from Definition 2.8 that $\mathbf{g}$ is defined by (2.8) with Λ being the identity matrix. Thus for any $\zeta = a^i * E_i \in TX$ and $\xi = (E_j)^t * b^j \in \tilde{T}X$ with $a_i, b_j \in \mathcal{A}$,

$$\mathbf{g}(\zeta, \xi) = a^i * E_i * (E_j)^t * b^j = a^i * g_{ij} * b^j.$$

This completes the proof. □

Let us now equip the left and right tangent bundles with the *canonical connections* given by $\omega_i = -\tilde{\omega}_i = -\partial_i e$, and denote the corresponding covariant derivatives by

$$\nabla_i : TX \longrightarrow TX, \quad \tilde{\nabla}_i : \tilde{T}X \longrightarrow \tilde{T}X.$$

In principle, one can take arbitrary connections for the tangent bundles, but we shall not allow this option in this paper.

The following elements of $\mathcal{A}$ are defined in [13],

$$ {}_c\Gamma_{ijl} = \frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ji}\right), \qquad \Upsilon_{ijl} = \frac{1}{2}\left(\partial_i(E_j) * (E_l)^t - E_l * \partial_i(E_j)^t\right),$$
$$\Gamma_{ijl} = {}_c\Gamma_{ijl} + \Upsilon_{ijl}, \qquad \tilde{\Gamma}_{ijl} = {}_c\Gamma_{ijl} - \Upsilon_{ijl},$$

where Υ_{ijk} was referred to as the noncommutative torsion. Set [13]

$$\Gamma_{ij}^{k} = \Gamma_{ijl} * g^{lk}, \quad \tilde{\Gamma}_{ij}^{k} = g^{kl} * \tilde{\Gamma}_{ijl}. \tag{3.3}$$

Then we have the following result.

Lemma 3.4.

$$\nabla_i E_j = \Gamma_{ij}^{k} * E_k, \qquad \tilde{\nabla}_i \tilde{E}^j = -\tilde{E}^k * \Gamma_{ki}^{j}. \tag{3.4}$$

Proof. Consider the first formula. Write $\partial_i e = \partial_i(\tilde{E}^k) * E_k + \tilde{E}^k * \partial_i E_k$. We have

$$\begin{aligned}\nabla_i E_j &= \partial_i E_j - E_j \partial_i * e\\ &= \partial_i E_j - \big(\partial_i(E_j * e) - \partial_i(E_j) * e\big)\\ &= \partial_i(E_j) * \tilde{E}^k * E_k.\end{aligned}$$

It was shown in [13] that $\Gamma_{ij}^{k} = \partial_i(E_j) * \tilde{E}^k$. This immediately leads to the first formula. The proof for the second formula is essentially the same. □

Note that the Lemma 3.4 can be re-stated as

$$\nabla_i E^j = -\tilde{\Gamma}_{ik}^{j} * E^k, \quad \tilde{\nabla}_i (E_j)^t = (E_k)^t * \tilde{\Gamma}_{ij}^{k}.$$

By using Lemma 2.9 and Lemma 3.4, we can easily prove the following result, which is equivalent to [13, Proposition 2.7].

Proposition 3.5. *The connections are metric compatible in the sense that*

$$\partial_i \mathbf{g}(\zeta, \xi) = \mathbf{g}(\nabla_i \zeta, \xi) + \mathbf{g}(\zeta, \tilde{\nabla}_i \xi), \qquad \forall \zeta \in TX,\ \xi \in \tilde{T}X. \tag{3.5}$$

For $\zeta = E_j$ and $\xi = (E_k)^t$, we obtain from (3.5) the following result for all i, j, k:

$$\partial_i g_{jk} - \Gamma_{ijk} - \tilde{\Gamma}_{ikj} = 0. \tag{3.6}$$

This formula is in fact equivalent to Proposition 3.5.

Define

$$R_{kij}^{l} = E_k * \mathcal{R}_{ij} * \tilde{E}^l, \quad \tilde{R}_{kij}^{l} = -g^{lq} * E_q * \mathcal{R}_{ij} * \tilde{E}^p * g_{pk}. \tag{3.7}$$

We can show by some lengthy calculations that

$$\begin{aligned}R_{kij}^{l} = &\quad -\partial_j \Gamma_{ik}^{l} - \Gamma_{ik}^{p} * \Gamma_{jp}^{l} + \partial_i \Gamma_{jk}^{l} + \Gamma_{jk}^{p} * \Gamma_{ip}^{l},\\ \tilde{R}_{kij}^{l} = &\quad -\partial_j \tilde{\Gamma}_{ik}^{l} - \tilde{\Gamma}_{jp}^{l} * \tilde{\Gamma}_{ik}^{p} + \partial_i \tilde{\Gamma}_{jk}^{l} + \tilde{\Gamma}_{ip}^{l} * \tilde{\Gamma}_{jk}^{p},\end{aligned} \tag{3.8}$$

which are the *Riemannian curvatures* of the left and right tangent bundles of the noncommutative space X given in [13, Lemma 2.12 and §4]. Therefore,

$$[\nabla_i, \nabla_j] E_k = R_{kij}^{l} * E_l, \qquad [\tilde{\nabla}_i, \tilde{\nabla}_j](E_k)^t = (E_l)^t * \tilde{R}_{kij}^{l}, \tag{3.9}$$

which were proved in [13]. Let $R_{lkij} = R_{kij}^{p} * g_{pl}$ and $\tilde{R}_{lkij} = -g_{kp} * \tilde{R}_{lij}^{p}$. By (2.9), $R_{klij} = \tilde{R}_{klij}$.

Definition 3.6. Let

$$R_{ij} = R^p_{ipj}, \qquad R = g^{ji} * R_{ij}, \tag{3.10}$$

and call them the *Ricci curvature* and *scalar curvature* of the noncommutative space respectively.

Then obviously

$$R_{ij} = -\mathbf{g}([\nabla_j, \nabla_l]E_i, \tilde{E}^l), \quad R = -\mathbf{g}([\nabla_i, \nabla_k]E^i, \tilde{E}^k). \tag{3.11}$$

3.2. **Second fundamental form.** In the theory of classical surfaces, the second fundamental form plays an important role. A similar notion exists for embedded noncommutative spaces.

Definition 3.7. We define the left and right *second fundamental forms* of the noncommutative surface X by

$$h_{ij} = \partial_i E_j - \Gamma^k_{ij} * E_k, \qquad \tilde{h}_{ij} = \partial_i E_j - E_k * \tilde{\Gamma}^k_{ij}. \tag{3.12}$$

It follows from equation (3.4) that

$$h_{ij} \bullet E_k = 0, \qquad E_k \bullet \tilde{h}_{ij} = 0. \tag{3.13}$$

The Riemann curvature $R_{lkij} = (\nabla_i \nabla_j - \nabla_j \nabla_i)E_k \bullet E_l$ can be expressed in terms of the second fundamental forms. Note that

$$R_{lkij} \;=\; \partial_j E_k \bullet \partial_i E_l - \partial_j E_k \bullet \tilde{\nabla}_i E_l - \partial_i E_k \bullet \partial_j E_l + \partial_i E_k \bullet \tilde{\nabla}_j E_l.$$

By Definition 3.7,

$$\begin{aligned} R_{lkij} &= \partial_j E_k \bullet \tilde{h}_{il} - \partial_i E_k \bullet \tilde{h}_{jl} \\ &= (\nabla_j E_k + h_{jk}) \bullet \tilde{h}_{il} - (\nabla_i E_k + h_{ik}) \bullet \tilde{h}_{jl}. \end{aligned}$$

Equation (3.13) immediately leads to the following result.

Lemma 3.8. *The following generalized Gauss equation holds:*

$$R_{lkij} \;=\; h_{jk} \bullet \tilde{h}_{il} - h_{ik} \bullet \tilde{h}_{jl}. \tag{3.14}$$

3.3. **Minkowski signature.** Now let us briefly comment on noncommutative spaces with Minkowski signatures embedded in higher dimensions [13]. Fix a diagonal $m \times m$ matrix $\eta = diag(-1, \ldots, -1, 1, \ldots, 1)$ with p of the diagonal entries being -1, and $q = m - p$ of them being 1. Given $X = \begin{pmatrix} X^1 & X^2 & \ldots & X^m \end{pmatrix}$ in ${}_l\mathcal{A}^m$, we define an $n \times n$ matrix $\mathbf{g} = (g_{ij})_{i,j=1,2,\ldots,n}$ with entries

$$g_{ij} = \sum_{\alpha=1}^{m} \partial_i X^\alpha * \eta_{\alpha\beta} * \partial_j X^\beta.$$

We call X a *noncommutative space* embedded in $\mathcal{A}^m$ if the matrix $\mathbf{g}$ is invertible. Denote its inverse matrix by (g^{ij}). Now the idempotent which gives rise to the left and right tangent bundles of X is given by

$$e = \eta(E_i)^t * g^{ij} * E_j,$$

which obviously satisfies $E_i * e = E_i$ for all i. The fibre metric of Definition 2.9 yields a metric on the embedded noncommutative surface X.

4. Elementary examples

We present several simple examples of noncommutative embedded spaces to illustrate the general theory developed in earlier sections. We shall mainly discuss the metric, (canonical) connection and Riemannian curvature for each noncommutative embedded space. However, in Section 4.4, we examine the tangent bundle as a projective module over the noncommutative algebra of functions by explicitly constructing the corresponding idempotent.

4.1. **Noncommutative sphere.** Let $U = (0,\pi)\times(0,2\pi)$, and we write θ and ϕ for t_1 and t_2 respectively. Let $X(\theta,\phi) = (X^1(\theta,\phi), X^2(\theta,\phi), X^3(\theta,\phi))$ be given by

$$X(\theta,\phi) = \left(\frac{\sin\theta\cos\phi}{\cosh \bar{h}}, \frac{\sin\theta\sin\phi}{\cosh \bar{h}}, \frac{\sqrt{\cosh 2\bar{h}}\cos\theta}{\cosh \bar{h}}\right) \tag{4.1}$$

with the components being smooth functions in $(\theta,\phi)\in U$. It can be shown that X satisfies the following relation

$$X^1 * X^1 + X^2 * X^2 + X^3 * X^3 = 1. \tag{4.2}$$

Thus we may regard the noncommutative surface defined by X as an analogue of the sphere S^2. We shall denote it by $S^2_{\bar{h}}$ and refer to it as a *noncommutative sphere*. We have

$$\begin{aligned} E_1 &= \left(\frac{\cos\theta\cos\phi}{\cosh \bar{h}}, \frac{\cos\theta\sin\phi}{\cosh \bar{h}}, -\frac{\sqrt{\cosh 2\bar{h}}\sin\theta}{\cosh \bar{h}}\right),\\ E_2 &= \left(-\frac{\sin\theta\sin\phi}{\cosh \bar{h}}, \frac{\sin\theta\cos\phi}{\cosh \bar{h}}, 0\right). \end{aligned}$$

The components $g_{ij} = E_i \bullet E_j$ of the metric $\mathbf{g}$ on $S^2_{\bar{h}}$ can now be calculated, and we obtain

$$\begin{aligned} &g_{11} = 1, \quad g_{22} = \sin^2\theta - \frac{\sinh^2 \bar{h}}{\cosh^2 \bar{h}}\cos^2\theta,\\ &g_{12} = -g_{21} = \frac{\sinh \bar{h}}{\cosh \bar{h}}\left(\sin^2\theta - \cos^2\theta\right). \end{aligned}$$

The components of this metric commute with one another as they depend on θ only. Thus it makes sense to consider the usual determinant G of $\mathbf{g}$. We have

$$\begin{aligned}G &= \sin^2\theta + \tanh^2\bar{h}(\cos^2 2\theta - \cos^2\theta)\\ &= \sin^2\theta[1 + \tanh^2\bar{h}(1 - 4\cos^2\theta)].\end{aligned}$$

The inverse metric is given by

$$\begin{aligned}g^{11} &= \frac{\sin^2\theta - \tanh^2\bar{h}\cos^2\theta}{\sin^2\theta + \tanh^2\bar{h}(\cos^2 2\theta - \cos^2\theta)},\\ g^{22} &= \frac{1}{\sin^2\theta + \tanh^2\bar{h}(\cos^2 2\theta - \cos^2\theta)},\\ g^{12} &= -g^{21} = \frac{\tanh\bar{h}\cos 2\theta}{\sin^2\theta + \tanh^2\bar{h}(\cos^2 2\theta - \cos^2\theta)}.\end{aligned}$$

Now we determine the connection and curvature tensor of the noncommutative sphere. The computations are quite lengthy, thus we only record the results here. For the Christoffel symbols, we have

$$\begin{aligned}&\Gamma_{111} = \tilde{\Gamma}_{111} = 0, && \Gamma_{112} = -\tilde{\Gamma}_{112} = \sin 2\theta\tanh\bar{h},\\ &\Gamma_{121} = -\tilde{\Gamma}_{121} = -\sin 2\theta\tanh\bar{h}, && \Gamma_{122} = \tilde{\Gamma}_{122} = \frac{1}{2}\sin 2\theta(1 + \tanh^2\bar{h}),\\ &\Gamma_{211} = -\tilde{\Gamma}_{211} = \sin 2\theta\tanh\bar{h}, && \Gamma_{212} = \tilde{\Gamma}_{212} = \frac{1}{2}\sin 2\theta(1 + \tanh^2\bar{h}),\\ &\Gamma_{221} = \tilde{\Gamma}_{221} = -\frac{1}{2}\sin 2\theta(1 + \tanh^2\bar{h}), && \Gamma_{222} = -\tilde{\Gamma}_{222} = \sin 2\theta\tanh\bar{h}.\end{aligned}$$

Note that $\Gamma_{112} \neq \tilde{\Gamma}_{112}$. We now find the asymptotic expansions of the curvature tensors with respect to h:

$$\begin{aligned}R_{1112} &= 2\bar{h} + (\frac{10}{3} + 4\cos 2\theta)\bar{h}^3 + O(\bar{h}^4),\\ R_{2112} &= -\sin^2\theta - \frac{1}{2}(4 + \cos 2\theta - \cos 4\theta)\bar{h}^2 + O(\bar{h}^4),\\ R_{1212} &= \sin^2\theta + \frac{1}{2}(4 + \cos 2\theta - \cos 4\theta)\bar{h}^2 + O(\bar{h}^4),\\ R_{2212} &= -2\sin^2\theta\bar{h} - (\frac{5}{3} + \frac{4}{3}\cos 2\theta - 4\cos 4\theta)\bar{h}^3 + O(\bar{h}^4).\end{aligned}$$

We can also compute asymptotic expansions of the Ricci curvature tensor

$$\begin{aligned}
R_{11} =& 1+(6+4\cos 2\theta)\bar{h}^2+O(\bar{h}^4),\\
R_{21} =& (2-\cos 2\theta)\bar{h}+\frac{1}{3}(16+19\cos 2\theta-6\cos 4\theta)\bar{h}^3+O(\bar{h}^4),\\
R_{12} =& (2+\cos 2\theta)\bar{h}+\frac{1}{3}(16+29\cos 2\theta+6\cos 4\theta)\bar{h}^3+O(\bar{h}^4),\\
R_{22} =& \sin^2\theta+\frac{1}{2}(3+5\cos 2\theta-2\cos 4\theta)\bar{h}^2+O(\bar{h}^4),
\end{aligned}$$

and the scalar curvature

$$R = 2+4(3+4\cos 2\theta)\bar{h}^2+O(\bar{h}^4).$$

By setting $\bar{h}=0$, we obtain from the various curvatures of $S^2_{\bar{h}}$ the corresponding objects for the usual sphere S^2. This is a useful check that our computations above are accurate.

4.2. **Noncommutative torus.** This time we shall take $U=(0,2\pi)\times(0,2\pi)$, and denote a point in U by (θ,ϕ). Let $X(\theta,\phi)=(X^1(\theta,\phi),X^2(\theta,\phi),X^3(\theta,\phi))$ be given by

$$X(\theta,\phi) = ((a+\sin\theta)\cos\phi,(a+\sin\theta)\sin\phi,\cos\theta) \tag{4.3}$$

where $a>1$ is a constant. Classically X is the torus. When we extend scalars from $\mathbb{R}$ to $\mathbb{R}[[\bar{h}]]$ and impose the star product on the algebra of smooth functions, X gives rise to a noncommutative torus, which will be denoted by $T^2_{\bar{h}}$. We have

$$\begin{aligned}
E_1 &= (\cos\theta\cos\phi,\cos\theta\sin\phi,-\sin\theta),\\
E_2 &= (-(a+\sin\theta)\sin\phi,(a+\sin\theta)\cos\phi,0).
\end{aligned}$$

The components $g_{ij}=E_i\bullet E_j$ of the metric $\mathbf{g}$ on $T^2_{\bar{h}}$ take the form

$$\begin{aligned}
&g_{11} = 1+\sinh^2\bar{h}\cos 2\theta,\\
&g_{22} = (a+\cosh\bar{h}\sin\theta)^2-\sinh^2\bar{h}\cos^2\theta,\\
&g_{12} = -g_{21} = -\sinh\bar{h}\cosh\bar{h}\cos 2\theta+a\sinh\bar{h}\sin\theta.
\end{aligned}$$

As they depend only on θ, the components of the metric commute with one another. The inverse metric is given by

$$\begin{aligned}
&g^{11} = \frac{(a+\cosh\bar{h}\sin\theta)^2-\sinh^2\bar{h}\cos^2\theta}{G},\\
&g^{22} = \frac{1+\sinh^2\bar{h}\cos 2\theta}{G},\\
&g^{12} = -g^{21} = \frac{\sinh\bar{h}\cosh\bar{h}\cos 2\theta+a\sinh\bar{h}\sin\theta}{G},
\end{aligned}$$

where G is the usual determinant of $\mathbf{g}$ given by

$$G = (\sin\theta + a\cosh\bar{h})^2 - a^2\sin^2\theta\sinh^2\bar{h}.$$

Now we determine the curvature tensor of the noncommutative torus. The computations can be carried out in much the same way as in the case of the noncommutative sphere, and we merely record the results here. For the connection, we have

$$\begin{aligned}
\Gamma_{111} &= -\sin 2\theta\sinh^2\bar{h}, & \Gamma_{112} &= (a\cos\theta + \sin 2\theta\cosh\bar{h})\sinh\bar{h},\\
\Gamma_{121} &= -\sin 2\theta\sinh\bar{h}\cosh\bar{h}, & \Gamma_{122} &= a\cos\theta\cosh\bar{h} + \frac{1}{2}\sin 2\theta\cosh 2\bar{h},\\
\Gamma_{211} &= -\sin 2\theta\sinh\bar{h}\cosh\bar{h}, & \Gamma_{212} &= a\cos\theta\cosh\bar{h} + \frac{1}{2}\sin 2\theta\cosh 2\bar{h},\\
\Gamma_{221} &= -(a\cos\theta + \frac{1}{2}\sin 2\theta)\cosh\bar{h}, & \Gamma_{222} &= (2a\cos\theta + \sin 2\theta\cosh\bar{h})\sinh\bar{h}.
\end{aligned}$$

We can find the asymptotic expansions of the curvature tensors with respect to $\bar{h}$:

$$\begin{aligned}
R_{1112} =& \frac{2\sin\theta(1 + a\sin\theta)}{a + \sin\theta}\bar{h} + O(\bar{h}^3),\\
R_{2112} =& -\sin\theta(a + \sin\theta) + O(\bar{h}^2),\\
R_{1212} =& \sin\theta(a + \sin\theta) + O(\bar{h}^2),\\
R_{2212} =& -2\sin^2\theta(1 + a\sin\theta)\bar{h} + O(\bar{h}^3).
\end{aligned}$$

We can also compute asymptotic expansions of the Ricci curvature tensor

$$\begin{aligned}
R_{11} =& \frac{\sin\theta}{a + \sin\theta} + O(\bar{h}^2),\\
R_{21} =& -\frac{\sin\theta(-3a + 5a\cos\theta - (5 + 2a^2)\sin\theta + \sin 3\theta)}{2(a + \sin\theta)^2}\bar{h} + O(\bar{h}^3),\\
R_{12} =& \frac{\sin\theta(a + \cos 2\theta + a\sin\theta)}{a + \sin\theta}\bar{h} + O(\bar{h}^3),\\
R_{22} =& \sin\theta(a + \sin\theta) + O(\bar{h}^2),
\end{aligned}$$

and the scalar curvature

$$R = \frac{2\sin\theta}{a + \sin\theta} + O(\bar{h}^2).$$

By setting $\bar{h} = 0$, we obtain from the various curvatures of $T^2_{\bar{h}}$ the corresponding objects for the usual torus T^2.

4.3. **Noncommutative hyperboloid.** Another simple example is the noncommutative analogue of the hyperboloid described by $X = (x, y, \sqrt{1 + x^2 + y^2})$. One may also change the parametrization and consider instead

$$X(r,\phi) = (\sinh r\cos\phi, \sinh r\sin\phi, \cosh r) \tag{4.4}$$

on $U = (0,\infty)\times(0,2\pi)$, where a point in U is denoted by (r,ϕ). When we extend scalars from $\mathbb{R}$ to $\mathbb{R}[[h]]$ and impose the star product on the algebra of smooth functions (of $t_1 = r$ and $t_2 = \phi$), X gives rise to a noncommutative hyperboloid, which will be denoted by H^2_h. We have

$$
\begin{aligned}
E_1 &= (\cosh r\cos\phi, \cosh r\sin\phi, \sinh r),\\
E_2 &= (-\sinh r\sin\phi, \sinh r\cos\phi, 0).
\end{aligned}
$$

The components $g_{ij} = E_i \bullet E_j$ of the metric $\mathbf{g}$ on H^2_h take the form

$$
\begin{aligned}
&g_{11} = \cos^2\bar{h}\cosh 2r,\\
&g_{22} = \frac{1}{2}\left(-1+\cos 2\bar{h}\cosh 2r\right),\\
&g_{12} = -g_{21} = -\frac{1}{2}\sin 2\bar{h}\cosh 2r.
\end{aligned}
$$

As they depend only on r, the components of the metric commute with one another. The inverse metric is given by

$$
\begin{aligned}
&g^{11} = \frac{\sec^2\bar{h}}{2\sinh^2 r}\left(\cos 2\bar{h} - \frac{1}{\cosh 2r}\right),\\
&g^{22} = \frac{1}{\sinh^2 r},\\
&g^{12} = -g^{21} = \frac{\tan\bar{h}}{\sinh^2 r}.
\end{aligned}
$$

Now we determine the curvature tensor of the noncommutative hyperboloid. For the connection, we have

$$
\begin{aligned}
&\Gamma_{111} = \cos^2\bar{h}\sinh 2r, &&\Gamma_{112} = -\frac{1}{2}\sin 2\bar{h}\sinh 2r,\\
&\Gamma_{121} = \frac{1}{2}\sin 2\bar{h}\sinh 2r, &&\Gamma_{122} = \frac{1}{2}\cos 2\bar{h}\sinh 2r,\\
&\Gamma_{211} = \frac{1}{2}\sin 2\bar{h}\sinh 2r, &&\Gamma_{212} = \frac{1}{2}\cos 2\bar{h}\sinh 2r,\\
&\Gamma_{221} = -\frac{1}{2}\cos 2\bar{h}\sinh 2r, &&\Gamma_{222} = \frac{1}{2}\sin 2\bar{h}\sinh 2r.
\end{aligned}
$$

We can find the asymptotic expansions of the curvature tensors with respect to $\bar{h}$:

$$\begin{aligned}
R_{1112} =& \frac{2}{\cosh 2r}\bar{h} + O(\bar{h}^2),\\
R_{2112} =& -\frac{\sinh^2 r}{\cosh 2r} + O(\bar{h}^2),\\
R_{1212} =& \frac{\sinh^2 r}{\cosh 2r} + O(\bar{h}^2),\\
R_{2212} =& -\frac{\cosh 2r + \sinh^2 2r}{\cosh 2r}\bar{h} + O(\bar{h}^3).
\end{aligned}$$

We can also compute asymptotic expansions of the Ricci curvature tensor

$$\begin{aligned}
R_{11} =& \frac{1}{\cosh 2r} + O(\bar{h}^2),\\
R_{21} =& \frac{\coth^2 r(2\cosh 2r - 1)}{\cosh 2r}\bar{h} + O(\bar{h}^3),\\
R_{12} =& \frac{\cosh 2r + 2}{\cosh^2 2r}\bar{h} + O(\bar{h}^3),\\
R_{22} =& \frac{\sinh^2 r}{\cosh^2 2r} + O(\bar{h}^2),
\end{aligned}$$

and the scalar curvature

$$R = \frac{2}{\cosh^2 2r} + O(\bar{h}^2).$$

By setting $\bar{h} = 0$, we obtain from the various curvatures of $H^2_{\bar{h}}$ the corresponding objects for the usual hyperboloid H^2.

4.4. **A time slice of a quantised Schwarzschild spactime.** We analyze an embedded noncommutative surface of Euclidean signature arising from the quantisation of a time slice of the Schwarzschild spacetime. While the main purpose here is to illustrate how the general theory developed in previous sections works, the example is interesting in its own right.

Let us first specify the notation to be used in this section. Let $t^1 = r,\ \ t^2 = \theta$ and $t^3 = \phi$, with $r > 2m$, $\theta \in (0,\pi)$, and $\phi \in (0, 2\pi)$. We deform the algebra of functions in these variables by imposing the Moyal product defined by (2.1) with the following anti-symmetric matrix

$$(\theta_{ij})_{i,j=1}^3 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0 \end{pmatrix}.$$

Note that the functions depending only on the variable r are central in the Moyal algebra $\mathcal{A}$. We shall write the usual pointwise product of two functions f and $\mathbf{g}$ as fg, but write their Moyal product as $f * g$.

Consider $X = \begin{pmatrix} X^1 & X^2 & X^3 & X^4 \end{pmatrix}$ with

$$\begin{aligned} X^1 = f(r) \quad \text{with} \quad (f')^2 + 1 = \left(1 - \frac{2m}{r}\right)^{-1}, \\ X^2 = r\sin\theta\cos\phi, \quad X^3 = r\sin\theta\sin\phi, \quad X^4 = r\cos\theta. \end{aligned} \tag{4.5}$$

Simple calculations yield

$$\begin{aligned} E_1 &= \partial_r X = \begin{pmatrix} f' & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}, \\ E_2 &= \partial_\theta X = \begin{pmatrix} 0 & r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \end{pmatrix}, \\ E_3 &= \partial_\phi X = \begin{pmatrix} 0 & -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix}. \end{aligned}$$

Using these formulae, we obtain the following expressions for the components of the metric of the noncommutative surface X:

$$\begin{aligned} g_{11} &= \left(1 - \frac{2m}{r}\right)^{-1} \left[1 - \left(1 - \frac{2m}{r}\right)\cos(2\theta)\sinh^2 \bar{h}\right], \\ g_{12} &= g_{21} = r\sin(2\theta)\sinh^2 \bar{h}, \\ g_{22} &= r^2\left[1 + \cos(2\theta)\sinh^2 \bar{h}\right], \\ g_{23} &= -g_{32} = -r^2\cos(2\theta)\sinh \bar{h}\cosh \bar{h}, \\ g_{13} &= -g_{31} = -r\sin(2\theta)\sinh \bar{h}\cosh \bar{h}, \\ g_{33} &= r^2\left[\sin^2\theta - \cos(2\theta)\sinh^2 \bar{h}\right]. \end{aligned} \tag{4.6}$$

In the limit $\bar{h} \to 0$, we recover the spatial components of the Schwarzschild metric. Observe that the noncommutative surface still reflects the characteristics of the Schwarzschild spacetime in that there is a time slice of the Schwarzschild black hole with the event horizon at $r = 2m$.

Since the metric $\mathbf{g}$ depends on θ and r only, and the two variables commute, the inverse (g^{ij}) of the metric can be calculated in the usual way as in the commutative case. Now the components of the idempotent $e = (e_{ij}) = (E_i)^t * g^{ij} * E_j$ are given by

the following formulae:

$$\begin{aligned}
e_{11} =& \frac{2m}{r} + \frac{2m(2m-r)(2+\cos 2\theta)}{r^2}\bar{h}^2 + O(\bar{h}^3),\\
e_{12} =& \frac{m\cos\phi\sin\theta}{r\sqrt{\frac{m}{-4m+2r}}} - \frac{2m\cos\theta\sin\phi}{r\sqrt{\frac{m}{-4m+2r}}}\bar{h}\\
&+ \frac{m(4m+r+2m\cos 2\theta)\cos\phi\sin\theta}{r^2\sqrt{\frac{m}{-4m+2r}}}\bar{h}^2 + O(\bar{h}^3)\\
e_{13} =& \frac{m\sin\theta\sin\phi}{r\sqrt{\frac{m}{-4m+2r}}} + \frac{2m\cos\theta\cos\phi}{r\sqrt{\frac{m}{-4m+2r}}}\bar{h}\\
&+ \frac{m(4m+r+2m\cos 2\theta)\sin\theta\sin\phi}{r^2\sqrt{\frac{m}{-4m+2r}}}\bar{h}^2 + O(\bar{h}^3)\\
e_{14} =& \frac{m\cos\theta}{r\sqrt{\frac{m}{-4m+2r}}} + \frac{m\cos\theta(4m-r+2m\cos 2\theta)}{r^2\sqrt{\frac{m}{-4m+2r}}}\bar{h}^2 + O(\bar{h}^3)
\end{aligned}$$

$$\begin{aligned}
e_{21} =& \frac{m\cos\phi\sin\theta}{r\sqrt{\frac{m}{-4m+2r}}} + \frac{2m\cos\theta\sin\phi}{r\sqrt{\frac{m}{-4m+2r}}}\bar{h}\\
&+ \frac{m(4m+r+2m\cos 2\theta)\cos\phi\sin\theta}{r^2\sqrt{\frac{m}{-4m+2r}}}\bar{h}^2 + O(\bar{h}^3)\\
e_{22} =& 1 - \frac{2m\sin^2\theta\cos^2\phi}{r}\\
&+ \frac{m}{2r^2}\Big[2r + 2m\cos 4\theta\cos^2\phi - 6m\cos^2\phi\\
&+ 2\cos 2\theta(m+8r+(m-r)\cos 2\phi)\Big]\bar{h}^2 + O(\bar{h}^3)\\
e_{23} =& -\frac{m\sin^2\theta\sin 2\phi}{r} - \frac{3m\sin 2\theta}{r}\bar{h}\\
&+ \frac{m(2(m-r)\cos 2\theta + m(-3+\cos 4\theta))\sin 2\phi}{2r^2}\bar{h}^2 + O(\bar{h}^3)\\
e_{24} =& \frac{-2m\cos\theta\cos\phi\sin\theta}{r} - \frac{m(1+3\cos 2\theta)\sin\phi}{r}\bar{h}\\
&- \frac{m(8m+5r+4m\cos 2\theta)\cos\phi\sin 2\theta}{2r^2}\bar{h}^2 + O(\bar{h}^3)
\end{aligned}$$

$$
\begin{aligned}
e_{31} =& \frac{m\sin\theta\sin\phi}{r\sqrt{\frac{m}{-4m+2r}}} - \frac{2m\cos\theta\cos\phi}{r\sqrt{\frac{m}{-4m+2r}}}\hbar \\
&+ \frac{m(4m+r+2m\cos 2\theta)\sin\theta\sin\phi}{r^2\sqrt{\frac{m}{-4m+2r}}}\hbar^2 + O(\hbar^3) \\
e_{32} =& -\frac{m\sin^2\theta\sin 2\phi}{r} + \frac{3m\sin 2\theta}{r}\hbar \\
&+ \frac{m(2(m-r)\cos 2\theta + m(-3+\cos 4\theta))\sin 2\phi}{2r^2}\hbar^2 + O(\hbar^3) \\
e_{33} =& 1 - \frac{2m\sin^2\theta\sin^2\phi}{r} \\
&+ \frac{m}{2r^2}\Big[2r + 2m\cos 4\theta\sin^2\phi - 6m\sin^2\phi \\
&+ 2\cos 2\theta(m+8r-(m-r)\cos 2\phi)\Big]\hbar^2 + O(\hbar^3) \\
e_{34} =& \frac{-2m\cos\theta\sin\theta\sin\phi}{r} + \frac{m(1+3\cos 2\theta)\cos\phi}{r}\hbar \\
&- \frac{m(8m+5r+4m\cos 2\theta)\sin 2\theta\sin\phi}{2r^2}\hbar^2 + O(\hbar^3)
\end{aligned}
$$

$$
\begin{aligned}
e_{41} =& \frac{m\cos\theta}{r\sqrt{\frac{m}{-4m+2r}}} + \frac{m\cos\theta(4m-r+2m\cos 2\theta)}{r^2\sqrt{\frac{m}{-4m+2r}}}\hbar^2 + O(\hbar^3) \\
e_{42} =& \frac{-2m\cos\theta\cos\phi\sin\theta}{r} + \frac{m(1+3\cos 2\theta)\sin\phi}{r}\hbar \\
&- \frac{m(8m+5r+4m\cos 2\theta)\cos\phi\sin 2\theta}{2r^2}\hbar^2 + O(\hbar^3) \\
e_{43} =& \frac{-2m\cos\theta\sin\theta\sin\phi}{r} - \frac{m(1+3\cos 2\theta)\cos\phi}{r}\hbar \\
&- \frac{m(8m+5r+4m\cos 2\theta)\sin 2\theta\sin\phi}{2r^2}\hbar^2 + O(\hbar^3) \\
e_{44} =& 1 - \frac{2m\cos^2\theta}{r} + \frac{4m\cos^2\theta(-2m+r-m\cos 2\theta)}{r^2}\hbar^2 + O(\hbar^3)
\end{aligned}
$$

Here we refrain from presenting the result of the Mathematica computation for the curvature $\mathcal{R}_{ij} = -[\partial_i e, \partial_j e]$, which is very complicated and not terribly illuminating. A detailed analysis of a quantised Schwarzschild spacetime will be given in Section 7.1.

5. General coordinate transformations

We now return to the general setting of Section 2 to investigate "general coordinate transformations". Our treatment follows closely [13, §V] and makes use of general ideas of [34, 27, 43]. We should point out that the material presented is part of an attempt of ours to develop a notion of "general covariance" in the noncommutative setting. This is an important matter which deserves a thorough investigation. We hope that the work presented here will prompt further studies.

Let $(\mathcal{A}, \mu)$ be a Moyal algebra of smooth functions on the open region U of $\mathbb{R}^n$ with coordinate t. This algebra is defined with respect to a constant skew symmetric matrix $\theta = (\theta_{ij})$. Let $\Phi : U \longrightarrow U$ be a diffeomorphism of U in the classical sense. We denote

$$u^i = \Phi^i(t),$$

and refer to this as a *general coordinate transformation* of U.

Denote by $\mathcal{A}_u$ the sets of smooth functions of $u = (u^1, u^2, \ldots, u^n)$. The map Φ induces an $\mathbb{R}[[\bar{h}]]$-module isomorphism $\phi = \Phi^* : \mathcal{A}_u \longrightarrow \mathcal{A}$ defined for any function $f \in \mathcal{A}_u$ by

$$\phi(f)(t) = f(\Phi(t)).$$

We define the $\mathbb{R}[[\bar{h}]]$-bilinear map

$$\mu_u : \mathcal{A}_u \otimes \mathcal{A}_u \longrightarrow \mathcal{A}_u, \quad \mu_u(f, g) = \phi^{-1}\mu_t(\phi(f), \phi(g)).$$

Then it is well-known [34] that μ_u is associative. Therefore, we have the associative algebra isomorphism

$$\phi : (\mathcal{A}_u, \mu_u) \xrightarrow{\sim} (\mathcal{A}_t, \mu_t).$$

We say that the two associative algebras are *gauge equivalent* by adopting the terminology of [27].

Following [13], we define $\mathbb{R}[[\bar{h}]]$-linear operators

$$\partial_i^\phi := \phi^{-1} \circ \partial_i \circ \phi : \mathcal{A}_u \longrightarrow \mathcal{A}_u, \tag{5.1}$$

which have the following properties [13, Lemma 5.5]:

$$\partial_i^\phi \circ \partial_j^\phi - \partial_j^\phi \circ \partial_i^\phi = 0,$$
$$\partial_i^\phi \mu_u(f, g) = \mu_u(\partial_i^\phi(f), g) + \mu_u(f, \partial_i^\phi(g)), \quad \forall f, g \in \mathcal{A}_u,$$

where the second relation is the Leibniz rule for ∂_i^ϕ. Recall that this Leibniz rule played a crucial role in the construction of noncommutative spaces over $(\mathcal{A}_u, \mu_u)$ in [13].

We shall denote by $\mathbf{M}_m(\mathcal{A}_u)$ the set of $m \times m$-matrices with entries in $\mathcal{A}_u$. The product of two such matrices will be defined with respect to the multiplication μ_u of the algebra $(\mathcal{A}_u, \mu_u)$. Then ϕ^{-1} acting component wise gives rise to an algebra isomorphism from $\mathbf{M}_m(\mathcal{A})$ to $\mathbf{M}_m(\mathcal{A}_u)$, where matrix multiplication in $\mathbf{M}_m(\mathcal{A})$ is defined with respect to μ.

Since we need to deal with two different algebras $(\mathcal{A},\mu)$ and $(\mathcal{A}_u,\mu_u)$ simultaneously in this section, we write μ and the matrix multiplication defined with respect to it by $*$ as before, and use $*_u$ to denote μ_u and the matrix multiplication defined with respect to it.

Let $e\in\mathbf{M}_m(\mathcal{A})$ be an idempotent. There exists the corresponding finitely generated projective left (resp. right) $\mathcal{A}$-module $\mathcal{M}$ (resp. $\tilde{\mathcal{M}}$). Now $e_u:=\phi^{-1}(e)$ is an idempotent in $\mathbf{M}_m(\mathcal{A}_u)$, that is, $\phi^{-1}(e)*_u\phi^{-1}(e)=\phi^{-1}(e)$. Write $e_u=(\mathcal{E}_\alpha^\beta)_{\alpha,\beta=1,\dots,m}$. This idempotent gives rises to the left projective $\mathcal{A}_u$-module $\mathcal{M}_u$ and right projective $\mathcal{A}_u$-module $\tilde{\mathcal{M}}_u$, respectively defined by

$$\mathcal{M}_u=\left\{\left(a^\alpha *_u \mathcal{E}_\alpha^1 \quad a^\alpha *_u \mathcal{E}_\alpha^2 \quad \dots \quad a^\alpha *_u \mathcal{E}_\alpha^m\right)\middle| a^\alpha\in\mathcal{A}_u\right\},$$

$$\tilde{\mathcal{M}}_u=\left\{\begin{pmatrix}\mathcal{E}_1^\beta *_u b_\beta\\ \mathcal{E}_2^\beta *_u b_\beta\\ \vdots\\ \mathcal{E}_m^\beta *_u b_\beta\end{pmatrix}\middle| b^\beta\in\mathcal{A}_u\right\},$$

where $a^\alpha *_u \mathcal{E}_\alpha^\beta=\sum_\alpha \mu_u(a^\alpha,\mathcal{E}_\alpha^\beta)$ and $\mathcal{E}_\alpha^\beta *_u b_\beta=\sum_\beta\mu_u(\mathcal{E}_\alpha^\beta,b_\beta)$. Below we consider the left projective module only, as the right projective module may be treated similarly.

Assume that we have the left connection

$$\nabla_i:\mathcal{M}\longrightarrow\mathcal{M},\quad \nabla_i\zeta=\frac{\partial\zeta}{\partial t^i}+\zeta*\omega_i.$$

Let $\omega_i^u:=\phi^{-1}(\omega_i)$. We have the following result.

Theorem 5.1. (1) *The matrices ω_i^u satisfy the following relations in $\mathbf{M}_m(\mathcal{A}_u)$:*

$$e_u *_u \omega_i^u *_u (1-e_u)=-e_u *_u \partial_i^\phi e_u.$$

(2) *The operators ∇_i^ϕ ($i=1,2,\dots,n$) defined for all $\eta\in\mathcal{M}_u$ by*

$$\nabla_i^\phi\eta=\partial_i^\phi\eta+\eta *_u\omega_i^u$$

give rise to a connection on $\mathcal{M}_u$.

(3) *The curvature of the connection ∇_i^ϕ is given by*

$$\mathcal{R}_{ij}^u=\partial_i^\phi\omega_j^u-\partial_j^\phi\omega_i^u-\omega_i^u *_u\omega_j^u+\omega_j^u *_u\omega_i^u,$$

which is related to the curvature $\mathcal{R}_{ij}$ of $\mathcal{M}$ by

$$\mathcal{R}_{ij}^u=\phi^{-1}(\mathcal{R}_{ij}).$$

Proof. Note that $e_u *_u\omega_i^u *_u(1-e_u)=\phi^{-1}(e*\omega_i*(1-e))$. We also have $\partial_i^\phi e_u=\phi^{-1}(\frac{\partial e}{\partial t^i})$, which leads to $e_u *_u\partial_i^\phi e_u=\phi^{-1}(e*\phi(\partial_i^\phi e_u))=\phi^{-1}(e*\partial_i e)$. This proves part (1). Part (2) follows from part (1) and the Leibniz rule for ∂_i^ϕ. Straightforward calculations show that the curvature of the connection ∇_i^ϕ is given by $\mathcal{R}_{ij}^u=\partial_i^\phi\omega_j^u-$

$\partial^{\phi}_j \omega^u_i - \omega^u_i *_u \omega^u_j + \omega^u_j *_u \omega^u_i$. Now $\partial^{\phi}_i \omega^u_j = \phi^{-1}(\frac{\partial \omega_j}{\partial t^i})$, and $\omega^u_i *_u \omega^u_j - \omega^u_j *_u \omega^u_i = \phi^{-1}(\omega_i * \omega_j) - \phi^{-1}(\omega_j * \omega_i)$. Hence $\mathcal{R}^u_{ij} = \phi^{-1}(\mathcal{R}_{ij})$. □

Remark 5.2. One can recover the usual transformation rules of tensors under the diffeomorphism group from the commutative limit of Theorem 5.1 in a way similar to that in [13, §5.C].

6. Noncommutative Einstein field equations and exact solutions

6.1. **Noncommutative Einstein field equations.** Recall that in classical Riemannian geometry, the second Bianchi identity suggests the correct form of Einstein's equation. Let us make some analysis of this point here.

In Section 3, we introduced the Ricci curvature R_{ij} and scalar curvature R. Let

$$R^i_j = g^{ik} * R_{kj},$$

then the scalar curvature is $R = R^i_i$. Let us also introduce the following object:

$$\Theta^l_p := \mathbf{g}([\nabla_p, \nabla_i]E^i, \tilde{E}^l) = g^{ik} * R^l_{kpi}. \tag{6.1}$$

In the commutative case, Θ^l_p coincides with R^l_p, but it is no longer true in the present setting. However, note that

$$\Theta^l_l = g^{ik} * R^l_{kli} = g^{ik} * R_{ki} = R. \tag{6.2}$$

By first contracting the indices j and l in the second Bianchi identity, then raising the index k to i by multiplying the resulting identity by g^{ik} from the left and summing over i, we obtain the identity

$$\begin{aligned} 0 = \partial_p R \quad & -\partial_i R^i_p \quad +\mathbf{g}\left([\nabla_i, \nabla_l]\nabla_p E^i, \tilde{E}^l\right) \quad +\mathbf{g}\left([\nabla_l, \nabla_p]\nabla_i E^i, \tilde{E}^l\right) \\ & -\partial_l \Theta^l_p \quad +\mathbf{g}\left([\nabla_i, \nabla_l]E^i, \tilde{\nabla}_p \tilde{E}^l\right) \quad +\mathbf{g}\left([\nabla_p, \nabla_i]E^i, \tilde{\nabla}_l \tilde{E}^l\right) \\ & \qquad\quad +\mathbf{g}\left([\nabla_p, \nabla_i]\nabla_l E^i, \tilde{E}^l\right) \quad +\mathbf{g}\left([\nabla_l, \nabla_p]E^i, \tilde{\nabla}_i \tilde{E}^l\right). \end{aligned}$$

Let us denote the sum of the last two terms on the right-hand side by ϖ_p. Then

$$\varpi_p \quad = \quad g^{ik} * R^r_{kpl} * \Gamma^l_{ri} - \tilde{\Gamma}^i_{lr} * g^{rk} * R^l_{kpi}.$$

In the commutative case, ϖ_p vanishes identically for all p. However in the noncommutative setting, there is no reason to expect this to happen. Let us now define

$$\begin{aligned} R^i_{p;i} = &\quad \partial_i R^i_p - \tilde{\Gamma}^i_{pr} * R^r_i + \tilde{\Gamma}^i_{ir} * R^r_p, \\ \Theta^l_{p;l} = &\quad \partial_l \Theta^l_p - \Theta^r_l * \Gamma^l_{rp} + \Theta^r_p * \Gamma^l_{rl} - \varpi_p. \end{aligned} \tag{6.3}$$

Then the second Bianchi identity implies

$$R^i_{p;i} + \Theta^i_{p;i} - \partial_p R = 0. \tag{6.4}$$

The above discussions suggest that Einstein's equation no longer takes its usual form in the noncommutative setting. Instead, formulae (6.4) and (6.2) suggest that the following is a reasonable proposal for a noncommutative Einstein equation in the vacuum:

$$R^i_j + \Theta^i_j - \delta^i_j R \quad = \quad 0. \tag{6.5}$$

We have not been able to formulate a basic principle which enables us to *derive* (6.6). However, in the next section, we shall solve this equation to obtain a class of exact solutions. The existence of such solutions is evidence that it is a meaningful candidate for a noncommutative Einstein field equation.

We may extend (6.5) to include matter and dark energy. We propose the following equation,

$$R^i_j + \Theta^i_j - \delta^i_j R + 2\delta^i_j \Lambda = 2T^i_j, \tag{6.6}$$

where T^i_j is some generalized "energy-momentum tensor", and Λ is the cosmological constant. This reduces to the vacuum equation (6.5) when $T^i_j = 0$ and the cosmological constant vanishes. We hope to provide a mathematical justification for this proposal in future work, where the defining properties of T^i_j will also be specified.

6.2. **Exact solutions in the vacuum.** We now construct a class of exact solutions of the noncommutative vacuum Einstein field equations. The solutions are quantum deformed analogues of plane-fronted gravitational waves [7, 29, 51, 30, 50, 19].

Let (θ_{ij}) be an arbitrary constant skew symmetric 4×4 matrix, and endow the space of functions of the variables (x,y,u,v) with the Moyal product defined with respect to (θ_{ij}). We denote the resulting noncommutative algebra by $\mathcal{A}$.

Now we consider a noncommutative space X embedded in $\mathcal{A}^6$ by a map of the form

$$X = \left(x, y, \frac{Hu+u+v}{\sqrt{2}}, \frac{H-\frac{u^2}{2}}{\sqrt{2}}, \frac{Hu-u+v}{\sqrt{2}}, \frac{H-\frac{u^2}{2}}{\sqrt{2}}\right), \tag{6.7}$$

where, needless to say, the component functions are elements of $\mathcal{A}$. Here H is an unknown function, which we shall determine by requiring noncommutative space to be Einstein.

Let us take $\eta = diag(1,1,1,1,-1,-1)$, and construct the noncommutative metric $\mathbf{g}$ by using the formula (2.9) for this embedded noncommutative space. Denote

$$A = \theta_{yu}H_{xy} + \theta_{xu}H_{xx}, \quad B = \theta_{yu}H_{yy} + \theta_{xu}H_{xy},$$

where H_{xy} etc. are second order partial derivatives of H, and θ_{xy} etc. refer to components of the matrix θ. A very lengthy calculation yields the following result for the

noncommutative metric:

$$\mathbf{g} = \begin{pmatrix} 1 & 0 & -\bar{h}A & 0 \\ 0 & 1 & -\bar{h}B & 0 \\ \bar{h}A & \bar{h}B & 2H & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is useful to note that in the classical limit with all $\theta_{ij} = 0$, the matrix diagonalises to $diag(1, 1, H+\sqrt{1+H^2}, H-\sqrt{1+H^2})$, thus has Minkowski signature. Further tedious computations produce the following inverse metric:

$$\mathbf{g}^{-1} = \begin{pmatrix} 1 & 0 & 0 & \bar{h}A \\ 0 & 1 & 0 & \bar{h}B \\ 0 & 0 & 0 & 1 \\ -\bar{h}A & -\bar{h}B & 1 & g^{44} \end{pmatrix} \quad \text{with} \quad g^{44} = -\bar{h}^2(B*B+A*A) - 2H.$$

Using these formulae we can compute Γ_{ijk} and Γ_{ij}^k, the nonzero components of which are given below:

$$\begin{aligned}
\Gamma_{113} =& -\bar{h}(\theta_{yu}H_{xxy} + \theta_{xu}H_{xxx}), \\
\Gamma_{123} =& \Gamma_{213} = -\bar{h}(\theta_{yu}H_{xyy} + \theta_{xu}H_{xxy}), \\
\Gamma_{133} =& \Gamma_{313} = H_x - \bar{h}(\theta_{yu}H_{xyu} + \theta_{xu}H_{xxu}), \\
\Gamma_{223} =& -\bar{h}(\theta_{yu}H_{yyy} + \theta_{xu}H_{xyy}), \\
\Gamma_{233} =& \Gamma_{323} = H_y - \bar{h}(\theta_{yu}H_{yyu} + \theta_{xu}H_{xyu}) \\
\Gamma_{331} =& -H_x, \quad \Gamma_{332} = -H_y, \\
\Gamma_{333} =& H_u - \bar{h}(\theta_{yu}H_{yuu} + \theta_{xu}H_{xuu}); \\
\Gamma_{11}^4 =& -\bar{h}(\theta_{yu}H_{xxy} + \theta_{xu}H_{xxx}), \\
\Gamma_{12}^4 =& \Gamma_{21}^4 = -\bar{h}(\theta_{yu}H_{xyy} + \theta_{xu}H_{xxy}) \\
\Gamma_{13}^4 =& \Gamma_{31}^4 = H_x - \bar{h}(\theta_{yu}H_{xyu} + \theta_{xu}H_{xxu}) \\
\Gamma_{22}^4 =& -\bar{h}(\theta_{yu}H_{yyy} + \theta_{xu}H_{xyy}), \\
\Gamma_{23}^4 =& \Gamma_{32}^4 = H_y - \bar{h}(\theta_{yu}H_{yyu} + \theta_{xu}H_{xyu}), \\
\Gamma_{33}^1 =& -H_x, \quad \Gamma_{33}^2 = -H_y, \\
\Gamma_{33}^4 =& -H_x * \bar{h}(\theta_{yu}H_{xy} + \theta_{xu}H_{xx}) - H_y * \bar{h}(\theta_{yu}H_{yy} + \theta_{xu}\, H_{xy}) \\
& + H_u - \bar{h}(\theta_{yu}H_{yuu} + \theta_{xu}H_{xuu}).
\end{aligned}$$

Remarkably, explicit formulae for curvatures can also be obtained, even though the noncommutativity of the $*$-product complicates the computations enormously. We

have

$$\begin{array}{ll} R_{1313} = -R_{1331} = -H_{xx}, & R_{1323} = -R_{1332} = -H_{xy}, \\ R_{2313} = -R_{2331} = -H_{xy}, & R_{2323} = -R_{2332} = -H_{yy}, \\ R_{3113} = -R_{3131} = H_{xx}, & R_{3123} = -R_{3132} = H_{xy}, \\ R_{3213} = -R_{3231} = H_{xy}, & R_{3223} = -R_{3232} = H_{yy}, \\ R_{3331} = -R_{3313} = 0, & R_{3332} = -R_{3323} = 0. \end{array}$$

Thus the nonzero components of R_{ijk}^{l} are

$$\begin{array}{ll} R_{113}^{4} = -R_{131}^{4} = H_{xx}, & R_{123}^{4} = -R_{132}^{4} = H_{xy}, \\ R_{213}^{4} = -R_{231}^{4} = H_{xy}, & R_{223}^{4} = -R_{232}^{4} = H_{yy}, \\ R_{313}^{1} = -R_{331}^{1} = -H_{xx}, & R_{313}^{2} = -R_{331}^{2} = -H_{xy}, \\ R_{323}^{1} = -R_{332}^{1} = -H_{xy}, & R_{323}^{2} = -R_{332}^{2} = -H_{yy}, \end{array}$$
$$\begin{aligned} R_{313}^{4} = -R_{331}^{4} &= -H_{xx} * \bar{h}(\theta_{yu}H_{xy} + \theta_{xu}H_{xx}) \\ &\quad -H_{xy} * \bar{h}(\theta_{yu}H_{yy} + \theta_{xu}H_{xy}), \\ R_{323}^{4} = -R_{332}^{4} &= -H_{xy} * \bar{h}(\theta_{yu}H_{xy} + \theta_{xu}H_{xx}) \\ &\quad -H_{yy} * \bar{h}(\theta_{yu}H_{yy} + \theta_{xu}H_{xy}). \end{aligned}$$

From these formulae, we obtain the nonzero components of the Ricci curvature:

$$R_{3}^{4} = \Theta_{3}^{4} = -H_{xx} - H_{yy}. \tag{6.8}$$

Thus the noncommutative vacuum Einstein field equations (6.5) are satisfied if and only if the following equation holds:

$$H_{xx} + H_{yy} = 0. \tag{6.9}$$

Solutions of this linear equation for H exist in abundance. Each solution leads to an exact solution of the noncommutative vacuum Einstein field equations. If we set θ to zero, we recover from such a solution the plane-fronted gravitational wave [7, 29, 51, 30] in classical general relativity. Thus we shall call such a solution of (6.5) a *plane-fronted noncommutative gravitational wave*.

It is clear from (6.8) that plane-fronted noncommutative gravitational waves satisfy the additivity property. Explicitly, if the noncommutative metrics of

$$X^{(i)} = \left(x, y, \frac{H_i u + u + v}{\sqrt{2}}, \frac{H_i - \frac{u^2}{2}}{\sqrt{2}}, \frac{H_i u - u + v}{\sqrt{2}}, \frac{H_i - \frac{u^2}{2}}{\sqrt{2}}\right), \quad i = 1, 2,$$

are plane-fronted noncommutative gravitational waves, we let $H = H_1 + H_2$, and set

$$X = \left(x, y, \frac{Hu + u + v}{\sqrt{2}}, \frac{H - \frac{u^2}{2}}{\sqrt{2}}, \frac{Hu - u + v}{\sqrt{2}}, \frac{H - \frac{u^2}{2}}{\sqrt{2}}\right).$$

Then the noncommutative metric of X is also a plane-fronted noncommutative gravitational wave. This is a rather nontrivial fact since the noncommutative Einstein field equations are highly nonlinear in $\mathbf{g}$, and it is extremely rare to have this additivity property.

At this point, it is appropriate to point out that the embedding (6.7) is only used as a device for constructing the metric and the connection, from which the curvatures are derived. However, we should observe the power of embeddings in solving the noncommutative Einstein field equations. Without using the embedding (6.7), it would be very difficult to come up with elegant solutions like what we have obtained here.

7. QUANTUM SPACETIMES

In this section, we consider quantisations of several well known spacetimes. We first find a global embedding of a spacetime into some pseudo-Euclidean space, whose existence is guaranteed by theorems of Nash, Clarke and Greene [48, 18, 37]. Then we quantise the spacetime following the strategy of deformation quantisation [6, 43] by deforming [34] the algebra of functions in the pseudo-Euclidean space to the Moyal algebra. Through this mechanism, classical spacetime metrics will deform to "quantum" noncommutative metrics which acquire quantum fluctuations. In particular, certain anti-symmetric components arise in the deformed metrics, which involve the Planck constant and vanish in the classical limit.

7.1. **Quantum deformation of the Schwarzschild spacetime.** In this section, we investigate noncommutative analogues of the Schwarzschild spacetime using the general theory discussed in previous sections. Recall that the Schwarzschild spacetime has the following metric

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{7.1}$$

where $m = \frac{2GM}{c^2}$ is constant, with M interpreted as the total mass of the spacetime. In the formula for m, G is the Newton constant, and c is the speed of light. The Schwarzschild spacetime can be embedded into a flat space of 6-dimensions in the following two ways [40, 41, 33]:

(i). Kasner's embedding:

$$\begin{aligned}
&X^1 = \left(1-\frac{2m}{r}\right)^{\frac{1}{2}}\sin t, \quad X^2 = \left(1-\frac{2m}{r}\right)^{\frac{1}{2}}\cos t,\\
&X^3 = f(r), \quad (f')^2 + 1 = \left(1-\frac{2m}{r}\right)^{-1}\left(1+\frac{m^2}{r^4}\right),\\
&X^4 = r\sin\theta\cos\phi, \quad X^5 = r\sin\theta\sin\phi, \quad X^6 = r\cos\theta,
\end{aligned}$$

with the Schwarzschild metric given by

$$ds^2 = -\left(dX^1\right)^2 - \left(dX^2\right)^2 + \left(dX^3\right)^2 + \left(dX^4\right)^2 + \left(dX^5\right)^2 + \left(dX^6\right)^2$$

(ii). Fronsdal's embedding:

$$\begin{aligned}
&Y^1 = \left(1-\frac{2m}{r}\right)^{\frac{1}{2}} \sinh t, \quad Y^2 = \left(1-\frac{2m}{r}\right)^{\frac{1}{2}} \cosh t,\\
&Y^3 = f(r), \quad (f')^2+1 = \left(1-\frac{2m}{r}\right)^{-1}\left(1-\frac{m^2}{r^4}\right),\\
&Y^4 = r\sin\theta\cos\phi, \quad Y^5 = r\sin\theta\sin\phi, \quad Y^6 = r\cos\theta,
\end{aligned}$$

with the Schwarzschild metric given by

$$ds^2 = -\left(dY^1\right)^2 + \left(dY^2\right)^2 + \left(dY^3\right)^2 + \left(dY^4\right)^2 + \left(dY^5\right)^2 + \left(dY^6\right)^2.$$

Let us now construct a noncommutative analogue of the Schwarzschild spacetime. Denote $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$. We deform the algebra of functions in these variables by imposing on it the Moyal product defined by (2.1) with the following anti-symmetric matrix

$$\left(\theta_{\mu\nu}\right)_{\mu,\nu=0}^{3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{7.2}$$

Denote the resultant noncommutative algebra by $\mathcal{A}$. Note that in the present case that the nonzero components of the matrix $\left(\theta_{\mu\nu}\right)$ are dimensionless.

Now we regard the functions X^i, Y^i $(1 \le i \le 6)$ appearing in both Kasner's and Fronsdal's embeddings as elements of $\mathcal{A}$. For $\mu = 0, 1, 2, 3$, and $i = 1, 2, \ldots, 6$, let

$$\begin{aligned}
E_\mu^i &= \frac{\partial X^i}{\partial x^\mu}, \quad \text{for Kasner's embedding,}\\
E_\mu^i &= \frac{\partial Y^i}{\partial x^\mu}, \quad \text{for Fronsdal's embedding.}
\end{aligned} \tag{7.3}$$

Following the general theory of the last section, we define the metric and noncommutative torsion for the noncommutative Schwarzschild spacetime by,

(1). in the case of Kasner's embedding

$$\begin{aligned}
g_{\mu\nu} =& -E_\mu^1 * E_\nu^1 - E_\mu^2 * E_\nu^2 + \sum_{j=3}^{6} E_\mu^j * E_\nu^j,\\
\Upsilon_{\mu\nu\rho} =& \frac{1}{2}\left(-\partial_\mu E_\nu^1 * E_\rho^1 - \partial_\mu E_\nu^2 * E_\rho^2 + \sum_{j=3}^{6} \partial_\mu E_\nu^j * E_\rho^j\right)\\
&+\frac{1}{2}\left(-E_\rho^1 * \partial_\mu E_\nu^1 - E_\rho^2 * \partial_\mu E_\nu^2 + \sum_{j=3}^{6} E_\rho^j * \partial_\mu E_\nu^j\right);
\end{aligned} \tag{7.4}$$

(2). in the case of Fronsdal's embedding

$$
\begin{aligned}
g_{\mu\nu} &= -E^1_\mu * E^1_\nu + \sum_{j=2}^{6} E^j_\mu * E^j_\nu, \\
\Upsilon_{\mu\nu\rho} &= \frac{1}{2}\left(-\partial_\mu E^1_\nu * E^1_\rho + \sum_{j=2}^{6} \partial_\mu E^j_\nu * E^j_\rho\right) \\
&\quad + \frac{1}{2}\left(-E^1_\rho * \partial_\mu E^1_\nu + \sum_{j=2}^{6} E^j_\rho * \partial_\mu E^j_\nu\right).
\end{aligned}
\tag{7.5}
$$

Some lengthy but straightforward calculations show that the metrics and the noncommutative torsions are respectively equal in the two cases. Since the noncommutative torsion will not be used in later discussions, we shall not spell it out explicitly. However, we record the metric $\mathbf{g} = (g_{\mu\nu})$ of the quantum deformation of the Schwarzschild spacetime below:

$$
\begin{aligned}
g_{00} &= -\left(1 - \frac{2m}{r}\right), \\
g_{01} &= g_{10} = g_{02} = g_{20} = g_{03} = g_{30} = 0, \\
g_{11} &= \left(1 - \frac{2m}{r}\right)^{-1}\left[1 + \left(1 - \frac{2m}{r}\right)\left(\sin^2\theta - \cos^2\theta\right)\sinh^2\bar{h}\right], \\
g_{12} &= g_{21} = 2r\sin\theta\cos\theta\sinh^2\bar{h}, \\
g_{13} &= -g_{31} = -2r\sin\theta\cos\theta\sinh\bar{h}\cosh\bar{h}, \\
g_{22} &= r^2\left[1 - \left(\sin^2\theta - \cos^2\theta\right)\sinh^2\bar{h}\right], \\
g_{23} &= -g_{32} = r^2\left(\sin^2\theta - \cos^2\theta\right)\sinh\bar{h}\cosh\bar{h}, \\
g_{33} &= r^2\left[\sin^2\theta + \left(\sin^2\theta - \cos^2\theta\right)\sinh^2\bar{h}\right].
\end{aligned}
\tag{7.6}
$$

It is interesting to observe that the quantum deformation of the Schwarzschild metric (7.6) still has a black hole with the event horizon at $r = 2m$. The Hawking temperature and entropy of the black hole are respectively given by

$$
T = \frac{1}{2}\frac{dg_{00}}{dr}\Big|_{r=2m} = \frac{1}{4m}, \qquad S_{bh} = 4\pi m^2.
$$

They coincide with the temperature and entropy of the classical Schwarzschild black hole of mass M. However, the area of the event horizon of the noncommutative black hole receives corrections from the quantum deformation of the spacetime. Let $\bar{g} =$

$\begin{pmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{pmatrix}$. We have

$$\begin{aligned} A =& \iint_{\{r=2m\}} \sqrt{\det \bar{g}}\, d\theta d\phi \\ =& \iint_{\{r=2m\}} r^2 \sin\theta \sqrt{1 + \left(\sin^2\theta - \cos^2\theta\right)\sinh^2 \bar{h}}\, d\theta d\phi \\ =& 16\pi m^2 \left(1 - \frac{\bar{h}^2}{6} + O(\bar{h}^4)\right). \end{aligned}$$

This leads to the following relationship between the horizon area and entropy of the noncommutative black hole:

$$S_{bh} = \frac{A}{4}\left(1 + \frac{\bar{h}^2}{6} + O(\bar{h}^4)\right). \tag{7.7}$$

Let us now consider the Ricci and Θ-curvature of the deformed Schwarzschild metric. We have

$$\begin{aligned} R_0^1 =& R_0^2 = R_0^3 = R_1^0 = R_2^0 = R_3^0 = 0, \\ \Theta_0^1 =& \Theta_0^2 = \Theta_0^3 = \Theta_1^0 = \Theta_2^0 = \Theta_3^0 = 0, \\ R_0^0 =& \Theta_0^0 = -\frac{m\left[2m + 3r + 3(m+r)\cos 2\theta\right]}{r^4}\bar{h}^2 + O(\bar{h}^4), \\ R_1^1 =& \Theta_1^1 = \frac{m\left[-14m + 3r + (-11m + r)\cos 2\theta\right]}{r^4}\bar{h}^2 + O(\bar{h}^4), \\ R_1^2 =& \Theta_1^2 = \frac{2m\cos^2\theta\cot\theta}{r^4}\bar{h}^2 + O(\bar{h}^4), \\ R_1^3 =& -\Theta_1^3 = \frac{2m\cot\theta}{r^4}\bar{h} + O(\bar{h}^3), \\ R_2^1 =& \Theta_2^1 = \frac{5m(-2m + r)\sin 2\theta}{r^3}\bar{h}^2 + O(\bar{h}^4), \\ R_2^2 =& \Theta_2^2 = \frac{m\left[4(m+r) + (6m + 5r)\cos 2\theta\right]}{r^4}\bar{h}^2 + O(\bar{h}^4), \\ R_2^3 =& -\Theta_2^3 = \frac{4m}{r^3}\bar{h} + O(\bar{h}^3), \\ R_3^1 =& -\Theta_3^1 = \frac{m(2m - r)\sin 2\theta}{r^3}\bar{h} + O(\bar{h}^3). \\ R_3^2 =& -\Theta_3^2 = \frac{4m\cos^2\theta}{r^3}\bar{h} + O(\bar{h}^3). \\ R_3^3 =& \Theta_3^3 = \frac{m\left[-8m + 8r + (-6m + 9r)\cos 2\theta\right]}{r^4}\bar{h}^2 + O(\bar{h}^4). \end{aligned}$$

Note that $R^i_i = \Theta^i_i$ for all i, and $R^j_i = -\Theta^j_i$ if $i \neq j$. Let us write

$$
\begin{aligned}
R^i_j &= R^i_{j(0)} + \bar{h} R^i_{j(1)} + \bar{h}^2 R^i_{j(2)} + \dots, \\
\Theta^i_j &= \Theta^i_{j(0)} + \bar{h}\Theta^i_{j(1)} + \bar{h}^2 \Theta^i_{j(2)} + \dots.
\end{aligned}
\tag{7.8}
$$

Then the formulae for R^i_j and Θ^i_j show that

$$R^i_{j(0)} = \Theta^i_{j(0)}, \quad R^i_{j(1)} = -\Theta^i_{j(1)}, \quad R^i_{j(2)} = \Theta^i_{j(2)}.$$

Naively generalizing the Einstein tensor $R^i_j - \frac{1}{2}\delta^i_j R$ to the noncommutative setting, one ends up with a quantity that does not vanish at order $\bar{h}$, as can be easily shown using the above results. However,

$$R^i_j + \Theta^i_j - \delta^i_j R = 0 + O(\bar{h}^2).$$

This indicates that the proposed noncommutative Einstein equation (6.6) captures some essence of the underlying symmetries in the noncommutative world.

Now the deformed Schwarzschild metric (7.6) satisfies the vacuum noncommutative Einstein equation (6.6) with $T^i_j = 0$ and $\Lambda = 0$ to first order in the deformation parameter. However, if we take into account higher order corrections in $\bar{h}$, the deformed Schwarzschild metric no longer satisfies the noncommutative Einstein equation in the vacuum. Instead, $R^i_j + \Theta^i_j - \delta^i_j R = T^i_j$ with T^i_j being of order $O(\bar{h}^2)$ and given by

$$
\begin{aligned}
T^1_0 &= T^2_0 = T^3_0 = T^0_1 = T^0_2 = T^0_3 = T^1_3 = T^2_3 = T^3_1 = T^3_2 = 0, \\
T^0_0 &= \frac{m\left[8m - 9r + (4m - 9r)\cos 2\theta\right]}{r^4}\bar{h}^2 + O(\bar{h}^4), \\
T^1_1 &= \frac{-m\left[4m + 3r + (4m + 5r)\cos 2\theta\right]}{r^4}\bar{h}^2 + O(\bar{h}^4), \\
T^2_1 &= \frac{2m\cos^2\theta\cot\theta}{r^4}\bar{h}^2 + O(\bar{h}^4), \\
T^1_2 &= \frac{5m\,(-2m + r)\sin 2\theta}{r^3}\bar{h}^2 + O(\bar{h}^4), \\
T^2_2 &= \frac{m\left[14m - 2r + (13m - r)\cos 2\theta\right]}{r^4}\bar{h}^2 + O(\bar{h}^4), \\
T^3_3 &= \frac{m\left[2\,(m + r) + (m + 3r)\cos 2\theta\right]}{r^4}\bar{h}^2 + O(\bar{h}^4).
\end{aligned}
\tag{7.9}
$$

A possible physical interpretation of the results is the following. We regard the $\bar{h}$ and higher order terms in the metric g_{ij} and associated curvature R_{ijkl} as arising from quantum effects of gravity. Then the T^j_i obtained in (7.9) should be interpreted as quantum corrections to the classical energy-momentum tensor.

7.2. **Quantum deformation of the Schwarzschild-de Sitter spacetime.** In this section, we investigate a noncommutative analogue of the Schwarzschild-de Sitter spacetime. Since the analysis is parallel to that on the quantum Schwarzschild spacetime, we shall only present the pertinent results.

Recall that the Schwarzschild-de Sitter spacetime has the following metric

$$
\begin{aligned}
ds^2 = &-\left(1-\frac{r^2}{l^2}-\frac{2m}{r}\right)dt^2+\left(1-\frac{r^2}{l^2}-\frac{2m}{r}\right)^{-1}dr^2 \\
&+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)
\end{aligned}
\tag{7.10}
$$

where $\frac{3}{l^2}=\Lambda>0$ is the cosmological constant, and m is related to the total mass of the spacetime through the same formula as in the Schwarzschild case. This spacetime can be embedded into a flat space of 6-dimensions in two different ways.

(i). Generalized Kasner embedding:

$$
\begin{aligned}
&X^1=\left(1-\frac{r^2}{l^2}-\frac{2m}{r}\right)^{\frac{1}{2}}\sin t, \quad X^2=\left(1-\frac{r^2}{l^2}-\frac{2m}{r}\right)^{\frac{1}{2}}\cos t, \\
&X^3=f(r), \quad (f')^2+1=\left(1-\frac{r^2}{l^2}-\frac{2m}{r}\right)^{-1}\left[1+\left(\frac{m}{r^2}-\frac{r}{l^2}\right)^2\right], \\
&X^4=r\sin\theta\cos\phi, \quad X^5=r\sin\theta\sin\phi, \quad X^6=r\cos\theta,
\end{aligned}
$$

with the Schwarzschild-de Sitter metric given by

$$
ds^2=-\left(dX^1\right)^2-\left(dX^2\right)^2+\left(dX^3\right)^2+\left(dX^4\right)^2+\left(dX^5\right)^2+\left(dX^6\right)^2
$$

(ii). Generalized Fronsdal embedding:

$$
\begin{aligned}
&Y^1=\left(1-\frac{r^2}{l^2}-\frac{2m}{r}\right)^{\frac{1}{2}}\sinh t, \quad Y^2=\left(1-\frac{r^2}{l^2}-\frac{2m}{r}\right)^{\frac{1}{2}}\cosh t, \\
&Y^3=f(r), \quad (f')^2+1=\left(1-\frac{r^2}{l^2}-\frac{2m}{r}\right)^{-1}\left[1-\left(\frac{m}{r^2}-\frac{r}{l^2}\right)^2\right], \\
&Y^4=r\sin\theta\cos\phi, \quad Y^5=r\sin\theta\sin\phi, \quad Y^6=r\cos\theta,
\end{aligned}
$$

with the Schwarzschild-de Sitter metric given by

$$
ds^2=-\left(dY^1\right)^2+\left(dY^2\right)^2+\left(dY^3\right)^2+\left(dY^4\right)^2+\left(dY^5\right)^2+\left(dY^6\right)^2.
$$

Let us now construct a noncommutative analogue of the Schwarzschild-de Sitter spacetime. Denote $x^0=t$, $x^1=r$, $x^2=\theta$ and $x^3=\phi$. We deform the algebra of functions in these variables by imposing on it the Moyal product defined by (2.1) with the anti-symmetric matrix (7.2). Denote the resultant noncommutative algebra by $\mathcal{A}$.

Now we regard the functions X^i, Y^i $(1\leq i\leq 6)$ appearing in both the generalized Kasner embedding and the generalized Fronsdal embedding as elements of $\mathcal{A}$. Let E^i_μ

($\mu = 0,1,2,3$, and $i = 1,2,\ldots,6$) be defined by (7.3) but for the generalized Kasner and Fronsdal embeddings respectively. We also define the metric and noncommutative torsion for the noncommutative Schwarzschild-de Sitter spacetime by equations (7.4) and (7.5) for the generalized Kasner and Fronsdal embeddings respectively. As in the case of the noncommutative Schwarzschild spacetime, we can show that the metrics and the noncommutative torsions are respectively equal for the two embeddings. We record the metric $\mathbf{g} = (g_{\mu\nu})$ of the quantum deformation of the Schwarzschild-de Sitter spacetime below:

$$
\begin{aligned}
g_{00} =& -\left(1-\frac{r^2}{l^2}-\frac{2m}{r}\right),\\
g_{01} =& g_{10} = g_{02} = g_{20} = g_{03} = g_{30} = 0,\\
g_{11} =& \left(1-\frac{r^2}{l^2}-\frac{2m}{r}\right)^{-1} + \left(\sin^2\theta-\cos^2\theta\right)\sinh^2\bar{h},\\
g_{12} =& g_{21} = 2r\sin\theta\cos\theta\sinh^2\bar{h},\\
g_{13} =& -g_{31} = -2r\sin\theta\cos\theta\sinh\bar{h}\cosh\bar{h},\\
g_{22} =& r^2\left[1-\left(\sin^2\theta-\cos^2\theta\right)\sinh^2\bar{h}\right],\\
g_{23} =& -g_{32} = r^2\left(\sin^2\theta-\cos^2\theta\right)\sinh\bar{h}\cosh\bar{h},\\
g_{33} =& r^2\left[\sin^2\theta+\left(\sin^2\theta-\cos^2\theta\right)\sinh^2\bar{h}\right].
\end{aligned}
\tag{7.11}
$$

Let us now consider the Ricci and Θ curvatures of the deformed Schwarzschild metric. We have

$$
\begin{aligned}
R^1_0 =& R^2_0 = R^3_0 = R^0_1 = R^0_2 = R^0_3 = 0,\\
\Theta^1_0 =& \Theta^2_0 = \Theta^3_0 = \Theta^0_1 = \Theta^0_2 = \Theta^0_3 = 0,\\
R^0_0 =& \Theta^0_0 = \frac{3}{l^2} + \Big[l^2\left(10m-3r\right)r^3 + 10r^6 - l^4 m\left(2m+3r\right)\\
&-3\left\{-2l^2mr^3-2r^6+l^4m\left(m+r\right)\right\}\cos 2\theta\Big]\frac{\bar{h}^2}{l^4r^4} + O(\bar{h}^4),\\
R^1_1 =& \Theta^1_1 = \frac{3}{l^2} + \Big[l^2\left(16m-9r\right)r^3 + 16r^6 + l^4m\left(-14m+3r\right)\\
&+\left\{2l^2\left(5m-2r\right)r^3+10r^6+l^4m\left(-11m+r\right)\right\}\cos 2\theta\Big]\frac{\bar{h}^2}{l^4r^4} + O(\bar{h}^4),\\
R^2_1 =& \Theta^2_1 = \frac{2\left(l^2m-4r^3\right)\cos^2\theta\cot\theta}{l^2r^4}\bar{h}^2 + O(\bar{h}^4),\\
R^3_1 =& -\Theta^3_1 = \frac{2\left(l^2m-4r^3\right)\cot\theta}{l^2r^4}\bar{h} + O(\bar{h}^3),\\
R^1_2 =& \Theta^1_2 = -\frac{\left[l^2\left(2m-r\right)+r^3\right]\left(5l^2m+4r^3\right)\sin 2\theta}{l^4r^3}\bar{h}^2 + O(\bar{h}^4),
\end{aligned}
$$

$$
\begin{aligned}
R_2^2 =&\Theta_2^2 = \frac{3}{l^2} + \Big[l^2\left(22m-r\right)r^3 + 10r^6 + 4l^4m\left(m+r\right)\\
&+\Big\{6r^6 + l^2r^3\left(15m+4r\right) + l^4m\left(6m+5r\right)\Big\}\cos 2\theta\Big]\frac{\bar{h}^2}{l^4r^4} + O(\bar{h}^4),\\
R_2^3 =& -\Theta_2^3 = \left(\frac{8}{l^2} + \frac{4m}{r^3}\right)\bar{h} + O(\bar{h}^3),\\
R_3^1 =& -\Theta_3^1 = \frac{\left(l^2m - 4r^3\right)\left[l^2\left(2m-r\right)+r^3\right]\sin 2\theta}{l^4r^3}\bar{h} + O(\bar{h}^3),\\
R_3^2 =& -\Theta_3^2 = 4\left(\frac{2}{l^2} + \frac{m}{r^3}\right)\cos^2\theta\bar{h} + O(\bar{h}^3),\\
R_3^3 =&\Theta_3^3 = \frac{3}{l^2} + \Big[-8l^4m\left(m-r\right) + l^2\left(28m-5r\right)r^3 + 16r^6\\
&+3\Big\{7l^2mr^3 + 4r^6 + l^4m\left(-2m+3r\right)\Big\}\cos 2\theta\Big]\frac{\bar{h}^2}{l^4r^4} + O(\bar{h}^4).
\end{aligned}
$$

Note that if we expand R_i^j and Θ_i^j into power series in $\bar{h}$ in the form (7.8), we again have

$$
R^i_{j(0)} = \Theta^i_{j(0)}, \quad R^i_{j(1)} = -\Theta^i_{j(1)}, \quad R^i_{j(2)} = \Theta^i_{j(2)}.
$$

By using the above results one can easily show that the deformed Schwarzschild-de Sitter metric (7.11) satisfies the vacuum noncommutative Einstein equation (6.6) (with $T_j^i = 0$) to first order in the deformation parameter:

$$
R_j^i + \Theta_j^i - \delta_j^i R + \delta_j^i\frac{6}{l^2} = 0 + O(\bar{h}^2).
$$

Further analysing the deformed Schwarzschild-de Sitter metric, we note that $R_j^i + \Theta_j^i - \delta_j^i R + \delta_j^i\frac{6}{l^2} = T_j^i$ with T_j^i being of order $O(\bar{h}^2)$ and given by

$$
\begin{aligned}
T_0^1 =&T_0^2 = T_0^3 = T_1^0 = T_2^0 = T_3^0 = T_3^1 = T_3^2 = T_1^3 = T_2^3 = 0,\\
T_0^0 =& -\Big[2l^2\left(14m-3r\right)r^3 + 16r^6 + l^4m\left(-8m+9r\right)\\
&+\Big\{20l^2mr^3 + 11r^6 + l^4m\left(-4m+9r\right)\Big\}\cos 2\theta\Big]\frac{\bar{h}^2}{l^4r^4} + O(\bar{h}^4),\\
T_1^1 =& -\Big[22l^2mr^3 + 10r^6 + l^4m\left(4m+3r\right)\\
&+\Big\{7r^6 + 4l^2r^3\left(4m+r\right) + l^4m\left(4m+5r\right)\Big\}\cos 2\theta\Big]\frac{\bar{h}^2}{l^4r^4} + O(\bar{h}^4),\\
T_1^2 =&\frac{2\left(l^2m - 4r^3\right)\cos^2\theta\cot\theta}{l^2r^4}\bar{h}^2 + O(\bar{h}^4),
\end{aligned}
$$

$$
\begin{aligned}
T_2^1 =& -\frac{\left[l^2(2m-r)+r^3\right]\left(5l^2m+4r^3\right)\sin 2\theta}{l^4r^3}\bar{h}^2+O(\bar{h}^4),\\
T_2^2 =& -\Big[2\Big\{4l^2(2m-r)r^3+8r^6+l^4m(-7m+r)\Big\}\\
&+\Big\{l^2(11m-4r)r^3+11r^6+l^4m(-13m+r)\Big\}\cos 2\theta\Big]\frac{\bar{h}^2}{l^4r^4}+O(\bar{h}^4),\\
T_3^3 =& \Big[2\Big\{-5r^6+l^4m(m+r)+l^2r^3(-5m+2r)\Big\}\\
&+\Big\{-5l^2mr^3-5r^6+l^4m(m+3r)\Big\}\cos 2\theta\Big]\frac{\bar{h}^2}{l^4r^4}+O(\bar{h}^4).
\end{aligned}
$$

Similar to the case of the quantum Schwarzschild spacetime, one may regard this as quantum corrections to the energy-momentum tensor.

7.3. **Noncommutative gravitational collapse.** Gravitational collapse is one of the most dramatic phenomena in the universe. When the pressure is not sufficient to balance the gravitational attraction inside a star, the star undergoes sudden gravitational collapse possibly accompanied by a supernova explosion, reducing to a super dense object such as a neutron star or black hole.

In 1939, Oppenheimer and Snyder [49] investigated the collapse process of ideal spherically symmetric stars equipped with the Tolman metric [58]. When the energy-momentum of an ideal star is assumed to be given by perfect fluids, Tolman's metric allows the case of dust which has zero pressure. In the dust case, Oppenheimer and Snyder solved the Einstein field equations by further assuming that the energy density is constant. They showed that stars above the Tolman-Oppenheimer-Volkoff mass limit [49] (approximately three solar masses) would collapse into black holes for reasons given by Chandrasekhar. The work of Oppenheimer and Snyder also marked the beginning of the modern theory of black holes.

The Tolman metric studied in [49] can be written as

$$
ds^2=-dt^2+(1-ct)^{4/3}\left[dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\right] \tag{7.12}
$$

with $c=3r_0^{\frac{1}{2}}R_b^{-\frac{3}{2}}$, where r_0 is the gravitational radius and R_b is the radius of the star. One may examine the behaviour of the scalar curvature as time increases. When time approaches the value $1/c$, the scalar curvature goes to ∞, thus the radius of the stellar object reduces to zero. By the reasoning of [59], this indicates gravitational collapse. Obviously this only provides a snapshot, nevertheless, it enables one to gain some understanding of gravitational collapse.

In this section, we quantise the dust solutions [49] and study noncommutative gravitational collapse. Our method for quantisation is much the same as in previous sections.

The Tolman spacetime can be embedded into a 5-dimensional flat Minkowski spacetime via

$$
\begin{aligned}
X^1 =& \frac{9(1-ct)^{4/3}}{32c^2} + \left(\frac{r^2}{4}+1\right)(1-ct)^{2/3},\\
X^2 =& \frac{9(1-ct)^{4/3}}{32c^2} + \left(\frac{r^2}{4}-1\right)(1-ct)^{2/3},\\
X^3 =& (1-ct)^{2/3} r\cos\phi\sin\theta, \quad X^4 = (1-ct)^{2/3} r\sin\theta\sin\phi,\\
X^5 =& (1-ct)^{2/3} r\cos\theta.
\end{aligned} \tag{7.13}
$$

We deform the algebra of functions in the variables r, t, π and θ into a Moyal algebra $\mathcal{A}$ defined by the anti-symmetric matrix

$$
(\theta^{\mu\nu})_{\mu,\nu=0}^{3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{7.14}
$$

Now we consider the noncommutative geometry embedded in $\mathcal{A}^5$ by (7.13). The noncommutative metric of the embedded noncommutative geometry (defined in the standard way [13]) yields a quantum deformation of the metric (7.12):

$$
\begin{aligned}
g_{\mu\nu} =& -\partial_\mu X^1 * \partial_\nu X^1 + \partial_\mu X^2 * \partial_\nu X^2 + \partial_\mu X^3 * \partial_\nu X^3\\
&+ \partial_\mu X^4 * \partial_\nu X^4 + \partial_\mu X^5 * \partial_\nu X^5,
\end{aligned} \tag{7.15}
$$

which can be computed explicitly. We have

$$
\begin{aligned}
g_{11} =& -\frac{4c^2 r^2 \cos 2\theta \sinh^2 \bar{h}}{9(1-ct)^{2/3}} - 1,\\
g_{12} =& g_{21} = \frac{2}{3} cr(1-ct)^{1/3} \cos 2\theta \sinh^2 \bar{h},\\
g_{13} =& g_{31} = -\frac{4}{3} cr^2 (1-ct)^{1/3} \cos\theta \sin\theta \sinh^2 \bar{h},\\
g_{14} =& -g_{41} = \frac{1}{3} cr^2 (1-ct)^{1/3} \sin 2\theta \sinh 2\bar{h},\\
g_{22} =& (1-ct)^{4/3}\left(1-\cos 2\theta \sinh^2 \bar{h}\right),\\
g_{23} =& g_{32} = r(1-ct)^{4/3} \sin 2\theta \sinh^2 \bar{h},\\
g_{24} =& -g_{42} = -2r(1-ct)^{4/3} \cos\theta \cosh \bar{h} \sin\theta \sinh \bar{h},\\
g_{33} =& r^2 (1-ct)^{4/3}\left(\cos 2\theta \sinh^2 \bar{h} + 1\right),
\end{aligned}
$$

$$g_{34} = -g_{43} = -\frac{1}{2}r^2(1-ct)^{4/3}\cos 2\theta \sinh 2\bar{h},$$
$$g_{44} = -\frac{1}{2}r^2(1-ct)^{4/3}(\cos 2\theta \cosh 2\bar{h} - 1).$$

The noncommutative scalar curvature is given by

$$R = \frac{4c^2 \cosh^2 \bar{h}}{(1-ct)^{4/3}} \frac{C_1}{C^3} \tag{7.16}$$

where C and C_1 are the following functions

$$C = 9(1-ct)^{2/3}\cosh^4 \bar{h} - 2c^2r^2(2\cos 2\theta + \cosh 2\bar{h} + 3)\sinh^2 \bar{h},$$

$$\begin{aligned} C_1 = &-243(1-ct)^{4/3}\cosh^8 \bar{h} + 486(1-ct)^{4/3}\cosh^6 \bar{h} \\ &- 18c^2r^2(1-ct)^{2/3}(2\cos 2\theta - 3\cosh 2\bar{h} - 1)\sinh^2 \bar{h}\cosh^4 \bar{h} \\ &- 9c^2r^2(1-ct)^{2/3}\Big(52\cosh 2\bar{h} + 3\cosh 4\bar{h} \\ &+ \cos 2\theta(28\cosh 2\bar{h} + \cosh 4\bar{h} - 13) + 9\Big)\sinh^2 \bar{h}\cosh^2 \bar{h} \\ &+ 4c^4r^4\sinh^4 \bar{h}\Big(4\cos 2\theta(\cosh 2\bar{h} + 15)\sinh^2 \bar{h} \\ &+ 2\cos 4\theta(\cosh 2\bar{h} - 3) + 38\cosh 2\bar{h} + 3\cosh 4\bar{h} - 13\Big). \end{aligned}$$

Let us regard $\bar{h}$ as a real number and make the (physically realistic) assumption that $\bar{h}$ is positive but close to zero. Now if t is significantly smaller than $\frac{1}{c}$ compared to $\bar{h}$, that is, $\frac{1}{c} - t \gg \bar{h}$, both the noncommutative metric and noncommutative scalar curvature R are finite, and there is non-singularity in the noncommutative spacetime. Thus the stellar object described by the noncommutative geometry behaves much the same as the corresponding classical object.

When $t = t_* := \frac{1}{c}$, we have $R|_{t_*=\frac{1}{c}} = \infty$ and the radius of the stellar object reduces to zero. This is the time when gravitational collapse happens in the usual classical setting.

However, in the noncommutative case, singularities of the scalar curvature already appear before t_*. Indeed, when time reaches

$$\begin{aligned} t(r,\theta) &= \frac{1}{c} - \frac{\sqrt{8}}{27}c^2r^3(2\cos 2\theta + \cosh 2\bar{h} + 3)^{3/2}\frac{\sinh^3 \bar{h}}{\cosh^6 \bar{h}} \\ &\cong \frac{1}{c} - \frac{8}{27}c^2r^3(\cos 2\theta + 2)^{3/2}\bar{h}^3, \end{aligned}$$

C vanishes and $\frac{C_1}{(1-ct)^{4/3}}$ is finite of order 0 in $\bar{h}$. Thus the scalar curvature tends to infinity for all $t(r,\theta)$ and the noncommutative spacetime becomes singular. Therefore, gravitational collapse happens within a certain range of time because of the quantum

effects captured by the noncommutativity of spacetime. However, effect of noncommutativity only starts to appear at third order of $\hbar$.

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